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An Algorithm for the Electromagnetic Scattering Due to an Axially Symmetric Body with an Impedance Boundary Condition

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Let B be a body in R^3, and let S denote the boundary of B. The surface S is described by $S = \{(x, y, z)\colon (x^2 + y^2)^{1/2} = f(z),\ -1 \leqslant z \leqslant 1\}$, where f is an analytic function that is real and positive on $(-1, 1)$ and $f(\pm 1) = 0$. An algorithm is described for computing the scattered field due to a plane wave incident field, under Leontovich boundary conditions. The Galerkin method of solution used here leads to a block diagonal matrix involving $2M + 1$ blocks, each block being of order $2(2N + 1)$. If, e.g., $N = O(M^2)$, the computed scattered field is accurate to within an error bounded by $Ce^{-cN^{1/2}}$, where C and c are positive constants depending only on f.

1. INTRODUCTION AND SUMMARY

Let B be a bounded in R^3, having surface S which is given by

$$S = \{(x, y, z)\colon (x^2 + y^2)^{1/2} = f(z), -1 \leqslant z \leqslant 1\}, \tag{1.1}$$

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where f is an analytic function that is real and positive on $(-1, 1)$ and

$$C_1(1+z)^{\alpha_1}(1-z)^{\beta_1} \leqslant f(z) \leqslant C_2(1+z)^{\alpha_2}(1-z)^{\beta_2}, \qquad -1 \leqslant z \leqslant 1, \quad (1.2)$$

where C_j, $\alpha_j < 1$ and $\beta_j < 1$ are positive constants. In this paper we describe an algorithm for computing the field scattered from B due to an incident field $\bar{E}^0(\bar{r})$ of the form

$$\bar{E}^0(\bar{r}) = \bar{\varepsilon} e^{i\bar{k}_0 \cdot \bar{r}}, \tag{1.3}$$

where $\bar{\varepsilon}$ and $\bar{k}_0$ ($|\bar{k}_0| = k_0 = \omega/c = 2\pi/\lambda$) denote the polarization and propagation vectors, respectively, and $\bar{r} = x\hat{x} + y\hat{y} + z\hat{z}$, where $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the unit vectors pointing in the direction of the x, y and z axes, respectively.

Let the body B (resp. free space) be homogeneous, with permitivity ε (resp. ε_0), permeability μ (resp. μ_0) and conductivity σ (resp. σ_0), so that the refractive index of the body is

$$N = \left\{ \frac{\mu}{\mu_0} \left(\frac{\varepsilon}{\varepsilon_0} + \frac{i\sigma}{\omega\varepsilon_0} \right) \right\}^{1/2}, \tag{1.4}$$

where ω denotes the frequency of the incident field. We shall furthermore assume that

$$|N|\,|k_0|\,\rho \gg 1, \tag{1.5}$$

where ρ is the smallest radius of curvature of S. This assumption enables us to apply the Leontovich boundary conditions [15, 21]

$$(\hat{n} \times \bar{E}) \times \hat{n} = \eta Z \hat{n} \times \bar{H} \tag{1.6}$$

on the surface of the body, where

$$\eta = \mu/(\mu_0 N), \qquad Z = (\mu_0/\varepsilon_0)^{1/2} \tag{1.7}$$

and where $\hat{n}$ denotes the outward unit normal to S, and $\bar{E}$ and $\bar{H}$ denote the total electric and magnetic fields on S. Conditions (1.5) and (1.6) are satisfied automatically if the body B is perfectly conducting, i.e. if $\sigma = \infty$.

Condition (1.6) makes it possible to obtain a singular vector integral equation over S for the surface current $\bar{K}$ on S. In the present paper we describe an algorithm for solving this integral equation via the Galerkin method, using as basis functions

$$\psi_{mn}(z,\varphi) = e^{im\varphi} f^{1/2}(z)(1-z^2) \times \frac{\sin\left[\pi N^{1/2}\left\{\log\left(\frac{1+z}{1-z}\right) - \frac{n}{N^{1/2}}\right\}\right]}{\pi N^{1/2}\left\{\log\left(\frac{1+z}{1-z}\right) - \frac{n}{N^{1/2}}\right\}}$$

$$m = 0, \pm 1, \ldots, \pm N, \qquad n = 0, \pm 1, \ldots, \pm M. \tag{1.8}$$

These basis functions effectively handle singularities of $\bar{K}$ and $\partial\bar{K}/\partial z$ as a function of z, which occur at $z = \pm 1$; they are similarly very effective approximants in the ϕ variable, since the Fourier series of $\bar{E}^0$ (and therefore $\bar{K}$) converges very rapidly. The singularities occurring in the kernel of the integral equation for $\bar{K}$ are of the type $1/(z'-z)$ or $\log|z'-z|$ at $z = z'$, and of the type of $f^m(z)$, $m = 0, 1, \ldots$ at $z = \pm 1$. The first of these is effectively handled by substracting out the principal value. The remaining ones are effectively handled by means of the quadrature formula (see [22])

$$\int_{-1}^{1} g(t)\,dt \cong h \sum_{k=-N}^{N} \frac{2e^{kh}}{(1+e^{kh})^2}\, g\left(\frac{e^{kh}-1}{e^{kh}+1}\right) \tag{1.9}$$

after transforming the intervals $(-1, z')$ and $(z', 1)$ to $(-1, 1)$.

The integrations over S involve two variables, φ and z. While the integrations with respect to z must be carried out numerically, due to singularities of the kernel in the region of integration, the integrations with respect to φ are carried out explicitly, the results being expressed via hypergeometric functions. The hypergeometric functions have logarithmic singularities which were not present in the kernel; for this reason explicit integration and later evaluation of the hypergeometric function as described in the Appendix have an advantage over direct numerical integration, since any known direct numerical integration procedure such as the trapezoidal rule would poorly handle this type of singularity.

The use of the basis functions (1.8) thus leads to a block diagonal Galerkin system of equations, one system of order $2(2N+1)$ for each m. By forming $2M+1$ such blocks, and taking $N = M^2$, we arrive at an approximation $\bar{K}_N$ of $\bar{K}$ which is accurate to within an error $\bar{\varepsilon}$, where $|\bar{\varepsilon}| = O(e^{-cN^{1/2}})$ as $N \to \infty$ with $c > 0$ and independent of N. The use of (1.8) furthermore makes it possible to evaluate the scattered field $\bar{E}^s$ by means of simple one-dimensional trapezoidal rule integrations. The error $\bar{\delta}_N$ in our approximation $\bar{E}^s_N$ of the scattered field $\bar{E}^s$ satisfies the relation

$$|\bar{\delta}_N(\bar{r})| = |\bar{E}^s(\bar{r}) - \bar{E}^s_N(\bar{r})| = O(e^{-cN^{1/2}}) \qquad \text{as} \quad N \to \infty, \tag{1.10}$$

for all $\bar{r}$ on the exterior of B and not arbitrarily close to B.

The paper is organized as follows.

The above problem of computing $\bar{E}^s$ given $\bar{E}^0$ as in (1.3) and S as in (1.1) was studied in [20] for the perfectly conducting case and in [5] for the case as described above. The Galerkin approximating basis function used in [5, 20] are of the form $\psi_{mn} = \cos(m\varphi)\, S_n(z)$ and $\sin(m\varphi)\, S_n(z)$, where S_n is the linear spline which is zero at z_{n-1} and z_{n+1}, and 1 at z_n. Thus the resulting rate of convergence is $O(1/N^2)$ if f has no singularities at $z = \pm 1$ and $O(1/N^\alpha)$ if, e.g., $f(z) \sim C(1-z)^\alpha$ as $z \to 1$, where it is assumed that the interval $(-1, 1)$ is divided into N equal subintervals. In addition, the quadratures used in [5, 20] converge very slowly as a result of the singularities present in the integral equation. Furthermore, the expression for the gradient of $G = e^{ik_0|\bar{r}-\bar{r}'|}/(4\pi|\bar{r}-\bar{r}'|)$ obtained in [5, 20] is incorrect.

The algorithm of the present paper is being checked out on a computer for the case of a sphere of radius 1 for which the surface current $\bar{K}$ and the scattered field $\bar{E}^s$ can be expressed explicitly. Using $M = 4$ and $N = M^2 = 16$, we expect to compute $\bar{K}$ accurate to four significant figures, and for $r \geqslant 2$, $\bar{E}^s$ is also expected to be accurate to four significant figures.

In Section 2 we describe the geometry of the surface. In Section 3 we derive a representation on the surface S for the incident plane wave. Section 4 contains a derivation of the integral equation for the surface current, as well as an integral expression for the scattered field in terms of this surface current. In Section 5 we describe the basis functions to be used in the Galerkin method of Section 7. Section 6 contains an approximate representation of the incident electric field, in terms of the basis functions of Section 5. In Section 7 we derive the Galerkin equations for the surface current, and we describe a method of computing the coefficients of this system, and for solving this system. In Section 8 we describe a procedure for evaluating the scattered field. In Section 9 we discuss the rate convergence of the procedure. Appendix A contains a study of the functions G_m derived in Section 7 as well as their derivatives. The results of this appendix illustrate the type of singular behavior of the functions G_m and thus they dictate the type of approximate methods to be used in order to achieve high accuracy, and they simplify our proof of convergence.

The rate of convergence of the method of this paper, namely, $O(e^{-cN^{1/2}})$ using an approximation of the form

$$\sum_{m=-M}^{M} \sum_{n=-N}^{N} a_{mn} \theta_n(z)\, e^{im\varphi} \qquad (N = M^2) \tag{1.11}$$

for each component of the surface current, is optimal, in a certain sense. By the results of [25], given any approximation method of type (1.11) which is to converge for all f analytic on $(-1, 1)$ and satisfying (1.2), the resulting error of this method cannot converge to zero faster than $e^{-\gamma N^{1/2}}$, for some $\gamma > 0$.

Similar numerical methods have been considered [7, 8, 9, 12, 18] but it is believed that the order of convergence obtained is not as good as that demonstrated here. For further discussion of such methods see, for example, Andreson [2], whose proposed method considers a maximum period of 20 wavelengths. Barber and Yeh [3] and Waterman [27] have considered extended boundary methods, while Kennaugh [13] and Schultz *et al.* [19] have discussed other implementations using a product z, φ basis.

2. Geometry of the Surface

A point on the surface S is represented by

$$\bar{r} = f(z) \cos \varphi \hat{x} + f(z) \sin \varphi \hat{y} + z\hat{z}, \tag{2.1}$$

where $\hat{x}$, $\hat{y}$ and $\hat{z}$ denote the unit vectors in the direction of the x, y and z axes, respectively.

It is convenient to introduce three unit vectors on the surface, $\hat{n}$, $\hat{\varphi}$ and $\hat{t}$, where

$$\begin{aligned} \hat{n} &= \alpha(z) \cos \varphi \hat{x} + \alpha(z) \sin \varphi \hat{y} - f'(z)\, \alpha(z) \hat{z}, \\ \hat{\varphi} &= -\sin \varphi \hat{x} + \cos \varphi \hat{y}, \\ \hat{t} &= f'(z)\, \alpha(z) \cos \varphi \hat{x} + f'(z)\, \alpha(z) \sin \varphi \hat{y} + \alpha(z) \hat{z}, \end{aligned} \tag{2.2}$$

where

$$\alpha(z) = [1 + f'(z)^2]^{-1/2}. \tag{2.3}$$

The vector $\hat{n}$ is the unit normal to the surface, $\hat{\varphi}$ is the unit vector at $\bar{r}$, pointing in the direction of increasing φ, and $\hat{t}$ is the unit longitudinal vector, pointing in the direction of increasing arc length. Thus $\hat{n}$, $\hat{\varphi}$, $\hat{t}$ and $\hat{x}$, $\hat{y}$, $\hat{z}$ are related by means of the equations

$$\begin{pmatrix} \hat{n} \\ \hat{\varphi} \\ \hat{t} \end{pmatrix} = \begin{pmatrix} \alpha(z) \cos \varphi & \alpha(z) \sin \varphi & -f'(z)\, \alpha(z) \\ -\sin \varphi & \cos \varphi & 0 \\ f'(z)\, \alpha(z) \cos \varphi & f'(z)\, \alpha(z) \sin \varphi & \alpha(z) \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \tag{2.4}$$

and

$$\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \alpha(z) \cos \varphi & -\sin \varphi & f'(z)\, \alpha(z) \cos \varphi \\ \alpha(z) \sin \varphi & \cos \varphi & f'(z)\, \alpha(z) \sin \varphi \\ -f'(z)\, \alpha(z) & 0 & \alpha(z) \end{pmatrix} \begin{pmatrix} \hat{n} \\ \hat{\varphi} \\ \hat{t} \end{pmatrix} \tag{2.5}$$

3. The Incident Electric Field

Let the incident radiation be a plane wave, given by

$$E^0(\bar{r}) = \bar{\varepsilon} e^{i\bar{k}_0 \cdot \bar{r}}, \tag{3.1}$$

where $\bar{\varepsilon}$ points in the direction of polarization, and $\bar{k}_0$ is the propagation vector which satisfies the relations

$$\begin{aligned} k_0 \equiv |\bar{k}_0| = \frac{\omega}{c} = \frac{2\pi}{\lambda}, \\ \bar{k}_0 \cdot \bar{\varepsilon} = 0. \end{aligned} \tag{3.2}$$

We shall furthermore assume that $\bar{k}_0$ lies in the xz plane and makes an angle θ_0 with the z-axis. Thus

$$\bar{k}_0 = k_0 \sin\theta_0 \hat{x} + k_0 \cos\theta_0 \hat{z}. \tag{3.3}$$

It is convenient to set

$$\bar{\varepsilon} = a_1 \bar{\varepsilon}_1 + a_2 \bar{\varepsilon}_2, \tag{3.4}$$

where a_1 and a_2 are scalars, while

$$\bar{\varepsilon}_1 = \hat{y}, \qquad \bar{\varepsilon}_2 = -\hat{x}\cos\theta_0 + \hat{z}\sin\theta_0. \tag{3.5}$$

Borison [4] has shown that if $a_2 = 0$ (resp. $a_1 = 0$) then the backscattered electric field $E^s(\bar{r})$ is polarized only in the direction $\bar{\varepsilon}_1$ (resp. $\bar{\varepsilon}_2$).

Using (2.5) and (3.5) we can express $\bar{\varepsilon}$ in components of $\hat{\varphi}$, $\hat{t}$ and $\hat{n}$. We get

$$\begin{aligned} \bar{\varepsilon} = a_1[&\alpha(z) \sin\varphi \hat{n} + \cos\varphi \hat{\varphi} + \alpha(z) f'(z) \sin\varphi \hat{t}] \\ &+ a_2[-\alpha(z)(\cos\theta_0 \cos\varphi + \sin\theta_0 f'(z))\hat{n} + \cos\theta_0 \sin\varphi \hat{\varphi} \\ &- \alpha(z)(\cos\theta_0 \cos f'(z) - \sin\theta_0)\hat{t}]. \end{aligned} \tag{3.6}$$

Next, let us find the Fourier expansion of $E^0(\bar{r})$ on S. To this end we use the identity

$$e^{ix\cos\varphi} = \sum_{m=-\infty}^{\infty} i^m J_m(x)\, e^{im\varphi}, \tag{3.7}$$

where $J_m(x)$ denotes the Bessel function,

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2/4)^{m+n}}{n!\,(n+m)!}.$$

Using (2.1) and (3.3) we get

$$\bar{k}_0 \cdot \bar{r} = k_0 f(z) \sin\theta_0 \cos\varphi + k_0 z \cos\theta_0. \tag{3.8}$$

Hence combining (3.1), (3.7) and (3.8) we find that the incident field on S is given by

$$\bar{E}^0(\bar{r}) = \bar{\varepsilon} e^{ik_0 z\cos\theta_0} \sum_{m=-\infty}^{\infty} i^m J_m(k_0 f(z) \sin\theta_0)\, e^{im\varphi}, \tag{3.9}$$

where $\bar{\varepsilon}$ is given in (3.6). Combining (3.6) and (3.9) we get

$$\begin{aligned}\bar{E}^0(\bar{r}) = \{&[a_1\alpha(z) \sin\varphi - a_2\alpha(z)(\cos\theta_0 \cos\varphi + f'(z) \sin\theta_0)]\hat{n} \\ &+ [a_1 \cos\varphi + a_2 \cos\theta_0 \sin\varphi]\hat{\varphi} \\ &+ [a_1\alpha(z) f'(z) - a_2\alpha(z)(\cos\theta_0 \cos\varphi f'(z) - \sin\theta_0)]\hat{t}\} \\ &\cdot e^{ik_0 z\cos\theta_0} \sum_{m=-\infty}^{\infty} i^m J_m(k_0 f(z) \sin\theta_0)\, e^{im\varphi}.\end{aligned} \tag{3.10}$$

4. The Scattered Field and the Integral Equation for the Surface Current

The scattered field $\bar{E}^s = \bar{E}^s(\bar{r}')$ is given in terms of the total electric ($\bar{E}$) and magnetic ($\bar{H}$) fields on S by [26],

$$\bar{E}^s = -\int_S [i\omega\mu(\hat{n} \times \bar{H})G + (\hat{n} \times \bar{E}) \times \nabla G + (\hat{n} \cdot \bar{E}) \nabla G]\, dS, \tag{4.1}$$

where it is assumed that the field vectors $\bar{E}$ and $\bar{H}$ have time dependence of the factored form $e^{-i\omega t}$, and

$$G = G(\bar{r}, \bar{r}') = \frac{1}{4\pi} \frac{\exp\{ik_0 |\bar{r} - \bar{r}'|\}}{|\bar{r} - \bar{r}'|}. \tag{4.2}$$

The remaining vectors in the integrand (4.1) are expressed in terms of $\bar{r}$, and ∇ is expressed in terms of $\bar{r}$.

Let $\bar{K}$ denote the surface current on S, due to the fields $\bar{E}$ and $\bar{H}$. In view of the Leontovich boundary condition (1.6), $\bar{K}$ satisfies the relations

$$\begin{gathered}\bar{K} = -\hat{n} \times \bar{H}; \qquad \hat{n} \times \bar{E} = -\eta Z \hat{n} \times \bar{K}, \\ \sigma = -\varepsilon\hat{n} \cdot \bar{E} = \frac{1}{i\omega} \nabla \cdot \bar{K}.\end{gathered} \tag{4.3}$$

Our development of the integral equation for $\bar{K}$ follows that in [5, 6]. We include the derivation from [5] for this equation, for the sake of completeness.

Let $\bar{A}$ be a continuous vector field tangent to S. Then the following results are valid, if S satisfies the Lyapunov conditions [17, p. 90]:

(a) the integral

$$I(\bar{r}') \equiv \int_S \bar{A}(\bar{r})\, G(\bar{r}, \bar{r}')\, dS$$

is a continuous function of $\bar{r}'$ in R^3.

(b) As $\bar{r}' \to r_0' \in S$, the relations

$$n(\bar{r}_0) \times \lim_{\bar{r}' \to \bar{r}_0'} \int_S \bar{A}(\bar{r}) \times \nabla G(\bar{r}, \bar{r}')\, dS$$

$$= \pm \frac{1}{2} A(\bar{r}_0') + \int_S \hat{n}(\bar{r}_0) \times [\bar{A}(\bar{r}) \times \nabla G(\bar{r}_0, \bar{r})]\, dS$$

are satisfied, where the plus (resp. minus) sign corresponds to an approach from the outside (resp. inside) of B.

(c) The term [17, p. 95]

$$\hat{n}(\bar{r}') \cdot \int_S \bar{A}(\bar{r}) \times \nabla G(\bar{r}, \bar{r}')\, dS$$

is a continuous function of $\bar{r}'$ on S. The term

$$\bar{E}_3(\bar{r}') \equiv \int_S \hat{n}(\bar{r}) \cdot \bar{E}(\bar{r})\, \nabla G(\bar{r}, \bar{r}')\, dS$$

suffers a discontinuity on transition through S equal to $\hat{n} \cdot \Delta\bar{E}_3$, where $\Delta\bar{E}_3$ is the difference of the values outside and inside. Therefore, the third term in $\bar{E}^s$ does not affect the tangential component, but reduces the normal component of $\bar{E}$ to zero.

Since the total electric field $\bar{E} = \bar{E}^0 + \bar{E}^s$ is zero inside the scatterer, (4.1) yields

$$\lim_{\bar{r}' \to \bar{r}_0' \in S} \hat{n}(\bar{r}_0') \times \left\{ \int_S [i\omega\mu(\hat{n} \times \bar{H})G + (\hat{n} \times E) \times \nabla G + (\hat{n} \cdot \bar{E})\, \nabla G]\, dS - \bar{E}^0(r') \right\} = 0,$$

where the approach is from the inside.

Application of (b) gives

$$0 = \hat{n}(\bar{r}') \times \bar{E}^0(\bar{r}') + \tfrac{1}{2}\hat{n}(\bar{r}') \times \bar{E}(\bar{r}') - \hat{n}(\bar{r}') \times \int_S [i\omega\mu(\hat{n} \times \bar{H})G + (\hat{n} \times \bar{E}) \times \nabla G + (\hat{n} \cdot \bar{E}) \nabla G]\, dS. \tag{4.4}$$

Next, taking the limit as $\bar{r}' \to \bar{r}_0'$ from the outside of B in (4.1) and using the relation

$$\hat{n}(\bar{r}_0') \times \bar{E}(\bar{r}_0') = \hat{n}(\bar{r}_0') \times \{\bar{E}^0(\bar{r}_0') + E^s(\bar{r}_0')\}$$

we get

$$\hat{n} \times \bar{E} = \hat{n} \times \bar{E}^0 - \lim_{\bar{r}' \to \bar{r}_0'} \hat{n}(\bar{r}_0') \times \int_S [i\omega\mu(\hat{n} \times \bar{H})G] + (\hat{n} \times \bar{E}) \times \nabla G + (\hat{n} \cdot \bar{E}) \nabla G]\, dS.$$

Application of (b) yields

$$\tfrac{3}{2}\hat{n}(\bar{r}') \times \bar{E}(\bar{r}') = \hat{n}(\bar{r}') \times \bar{E}^0(\bar{r}) - \hat{n}(\bar{r}') \times \int_S [i\omega\mu(\hat{n} \times \bar{H})G + (\hat{n} \times \bar{E}) \times \nabla G + (\hat{n} \cdot \bar{E}) \nabla G]\, dS. \tag{4.5}$$

Adding (4.4) and (4.5), we get

$$\hat{n}(\bar{r}') \times \bar{E}(\bar{r}') + 2\hat{n}(\bar{r}') \times \int_S [i\omega\mu(\hat{n} \times \bar{H})G + (\hat{n} \times \bar{E}) \times \nabla G + (\hat{n} \cdot \bar{E}) \nabla G]\, dS = 2\hat{n}(\bar{r}') \times E^0(\bar{r}').$$

Definitions (4.3) now yield

$$-\eta Z \hat{n}(\bar{r}') \times \bar{K}(\bar{r}') + 2\hat{n}(\bar{r}') \times \int_S \left\{ -i\omega\mu \bar{K} G - \eta Z(\hat{n} \times \bar{K}) \times \nabla G - \frac{1}{i\omega\mu} (\nabla \cdot \bar{K}) \nabla G \right\} dS = 2\hat{n}(\bar{r}') \times \bar{E}^0(\bar{r}').$$

This equation can be written in the equivalent form

$$\left[\frac{1}{2} \eta Z \bar{K}(\bar{r}') + \int_S \left\{ i\omega\mu \bar{K} G + Z(\hat{n} \times \bar{K}) \times \nabla G + \frac{1}{i\omega\varepsilon} (\nabla \cdot \bar{K}) \nabla G \right\} dS + \bar{E}^0(\bar{r}') \right]_{\tan \bar{r}' \in S} = 0. \tag{4.6}$$

Using Eqs. (4.3), the scattered field $\bar{E}^s$ given by (4.1) is expressed in terms of $\bar{K}$ by means of the integral

$$\bar{E}^s = \int_S \left[i\omega\mu\bar{K}G + \eta Z(\hat{n} \times \bar{K}) \times \nabla G + \frac{1}{i\omega\varepsilon} (\nabla \cdot \bar{K}) \nabla G \right] dS. \tag{4.7}$$

5. The Basis Functions for Approximating Surface Current and Electric Field

$\mathbb{C}$ denotes the complex plane, let $d > 0$ and $d' > 1$, and let Ω_d and $A_{d'}$ be defined by

$$\begin{aligned} \Omega_d &= \{z \in \mathbb{C} : |\arg[(1+z)/(1-z)]| < d\}, \\ A_{d'} &= \{w \in \mathbb{C} : 1/d' < |w| < d'\} \end{aligned} \tag{5.1}$$

(see Figs. 5.1 and 5.2).

Let $H(\Omega_d)$ (resp. $H(A_{d'})$) denote the family of all functions g that are analytic in Ω_d (resp. $A_{d'}$) such that

$$\|g\|_{\Omega_d} \equiv \sup_{z \in \Omega_d} |g(z)| < \infty \qquad (\text{resp. } \|g\|_{A_{d'}} = \sup_{z \in A_{d'}} |g(z)| < \infty). \tag{5.2}$$

Let us set

$$v(z) = f(z)(1 - z^2), \tag{5.3}$$

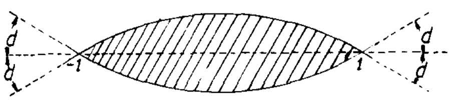

FIG. 5.1. The region Ω_d.

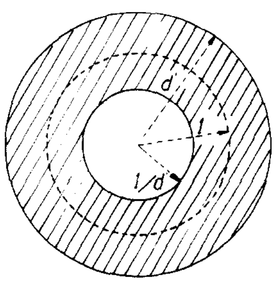

FIG. 5.2. The region $A_{d'}$.

where $f \in H(\Omega_d)$ satisfies (1.2), $f \neq 0$ in Ω_d. Let $H = H(d, d')$ denote the family of all functions $F = F(z, w)$ such that

(i) $F(z, e^{i\varphi})/v(z)$ belongs to $H(\Omega_d)$ as a function of z for all $\varphi \in [0, 2\pi]$;

(ii) $F(z, w)$ belongs to $H(A_{d'})$ as a function of w for all $z \in [-1, 1]$.

We define a norm on $H(d, d')$ by

$$\|F\|_H = \max\{\max_{\varphi \in [0,2\pi]} \|F\|_{\Omega_d}, \max_{z \in [-1,1]} \|F\|_{A_{d'}}\}. \tag{5.4}$$

If $F \in H(d, d')$, we approximate F on $\mathscr{S} = [-1, 1] \times [0, 2\pi]$ as follows,

$$F(z, e^{i\varphi}) \cong l_{MN}(z, \varphi) \equiv \sum_{m=-M}^{M} \sum_{n=-N}^{M} a_{mn} \psi_{mn}(z, \varphi), \tag{5.5}$$

where the a_{mn} are numbers and the ψ_{mn} are basis functions given by [23]

$$\begin{aligned} \psi_{mn}(z, \varphi) &\equiv e^{im\varphi} \theta_n(z), \qquad h > 0, \\ \theta_n(z) &\equiv f^{1/2}(z)(1 - z^2) \operatorname{sinc}\left[\frac{\omega(z) - nh}{h}\right], \\ \omega(z) &\equiv \log\left(\frac{1+z}{1-z}\right); \qquad \operatorname{sinc} x = \frac{\sin(\pi x)}{\pi x}; \end{aligned} \tag{5.6}$$

The numbers a_{mn} are given by

$$\begin{aligned} a_{mn} &= \frac{1}{2\pi f^{1/2}(z_n)(1 - z_n^2)} \int_0^{2\pi} F(z_n, e^{i\varphi})\, e^{-im\varphi}\, d\varphi, \\ z_n &= \tanh(nh)/2. \end{aligned} \tag{5.7}$$

Next, recalling the definition of f and relation (1.2), let us set

$$\gamma_2 = \tfrac{1}{2} \min(\alpha_2, \beta_2), \qquad \gamma = \min[(\pi d \gamma_2)^{1/2}, \log d']. \tag{5.8}$$

THEOREM 5.1. *Let $F \in H$, let $N = M^2$, and let $h = [\pi d/(\gamma N)]^{1/2}$. Then there exist constants C, C_1 and C_2 which are independent of N, such that*

$$\max_{(z,\varphi) \in S^*} |F(z, e^{i\varphi}) - l_{MN}(z, \varphi)| \leqslant C N^{1/2} e^{-\gamma N^{1/2}}, \tag{5.9}$$

$$\max_{(z,\varphi) \in S^*} \left| \frac{\partial}{\partial z} F(z, \varphi) - \frac{\partial}{\partial z} l_{MN}(z, \varphi) \right| \leqslant C_1 N e^{-\gamma N^{1/2}}, \tag{5.10}$$

$$\max_{(z,\varphi) \in S^*} \left| \frac{\partial}{\partial \varphi} F(z, \varphi) - \frac{\partial}{\partial \varphi} l_{MN}(z, \varphi) \right| \leqslant C_2 N^{1/2} e^{-\gamma N^{1/2}}, \tag{5.11}$$

where $S^* = [-1, 1] \times [0, 2\pi]$.

Proof. It is shown in [16] that if $g/\nu \in (\Omega_d)$, then by taking $h = [\pi d/(\gamma_2 N)]^{1/2}$ there are positive constants C' and C'' such that

$$\max_{z \in [-1,1]} \left| g(z) - \sum_{n=-N}^{N} \frac{f^{1/2}(z_n)}{\nu(z_n)} \theta_n(z) \right| \leqslant C' e^{-\nu N^{1/2}} \tag{5.12}$$

and

$$\max_{z \in [-1,1]} \left| g'(z) - \sum_{n=-N}^{N} \frac{f'(z_n)}{\nu(z_n)} \theta_n'(z) \right| \leqslant C'' N^{1/2} e^{-\nu N^{1/2}}. \tag{5.13}$$

Similarly, it follows from Cauchy's theorem, that if $G \in H(A_{d'})$, and if a_m is defined by

$$a_m = \frac{1}{2\pi} \int_0^{2\pi} G(e^{i\varphi})\, e^{-im\varphi}\, d\varphi, \qquad m = -M, -M+1, \dots, M, \tag{5.14}$$

then there are constants K' and K'' such that for all $\varphi \in [0, 2\pi]$,

$$\left| G(e^{i\varphi}) - \sum_{m=-M}^{M} a_m e^{im\varphi} \right| \leqslant K'(d')^{-M} \tag{5.15}$$

and

$$\left| \frac{\partial}{\partial \varphi} G(e^{i\varphi}) - \sum_{m=-M}^{M} i m a_m e^{im\varphi} \right| \leqslant K'' M (d')^{-M}. \tag{5.16}$$

On noting that $M = N^{1/2}$, inequality (5.9) is obtained if we use (5.11) and (5.15) in Theorem 6.2 in [24]. Similarly, (5.10) is a consequence of (5.13) and (5.15), while (5.11) follows from (5.12) and (5.16).

THEOREM 5.2 [16]. *If $\nu g/f^{1/2} \in H(\Omega_d)$, then there exists a constant K such that*

$$\left| \int_{-1}^{1} g(z)\, \theta_n(z)\, dz - h \frac{\nu(z_n)\, g(z_n)}{f^{1/2}(z_n)\, \omega'(z_n)} \right| \leqslant K e^{-\pi d/h}. \tag{5.17}$$

This theorem shows that if g is any function such that $\nu g/f^{1/2} \in H(\Omega_d)$, then for h sufficiently small, the sequence $\{\theta_n\}_{-\infty}^{\infty}$ may be considered to be an orthogonal sequence, for practical purposes, and g may be expanded with respect to this sequence. The coefficients of this expansion take on the very simple form

$$\frac{f^{1/2}(z_n)}{\nu(z_n)}\, g(z_n), \qquad z_n = \tanh(nh/2). \tag{5.18}$$

For purposes of numerical integration, we state the following.

THEOREM 5.3 [23]. *Let $g \in H(\Omega_d)$, and let $|g(z)| \leqslant C(1-z^2)^{\alpha-1}$ on $(-1, 1)$, where $C_1\alpha > 0$. If $N > 0$ is an integer, and $h = (2\pi d/\alpha N)^{1/2}$, then there exists a constant C', independent of N, such that*

$$\left| \int_{-1}^{1} g(z)\, dz - h \sum_{n=-N}^{N} \frac{g(z_n)}{\omega'(z_n)} \right| \leqslant C' e^{-(2\pi d\alpha N)^{1/2}}. \tag{5.19}$$

Finally, Theorem 5.4 which follows describes the accuracy of the midordinate rule used in Sections 7 and 8 for integrating periodic functions.

THEOREM 5.4. *Let $d' > 0$ and let $g \in H(A_{d'})$ be bounded on $A_{d'}$. Then there exists a constant C depending only on g such that for $M = 1, 2, 3,\ldots,$*

$$\left| \int_0^{2\pi} g(e^{i\varphi})\, d\varphi - \frac{2\pi}{M} \sum_{k=1}^{M} g(e^{(2k-1)\pi i/M}) \right| \leqslant C(d')^{-M}. \tag{5.20}$$

6. APPROXIMATION OF THE INCIDENT ELECTRIC FIELD

We shall make the approximation

$$\bar{E}^0(\bar{r}) \cong \bar{\mathscr{E}}^0(\bar{r}) \equiv \sum_{m=-M}^{M} \sum_{n=-N}^{N} [\alpha'_{mn}\hat{\varphi} + \beta'_{mn}\hat{t}]\, \psi_{mn}, \qquad \bar{r} \in S, \tag{6.1}$$

of the incident electric field, where ψ_{mn} are defined in Section 5. For the sake of convenience we shall use the notations

$$\begin{aligned} z_n &= \tanh(nh/2), \\ d_{mn} &= \frac{i^m \exp\{ik_0 z_n \cos\theta_0\}\, J_m(k_0 f(z_n)\sin\theta_0)}{\nu(z_n)}. \end{aligned} \tag{6.2}$$

As will be seen in Section 7, we will approximate a slightly altered field $\bar{J}^0 \equiv \nu \bar{E}^0$, where $\nu(z) = f(z)(1-z^2)$. Thus

$$\bar{J}^0(\bar{r}) \cong \bar{\mathscr{J}}^0 \equiv \sum_{m=-M}^{M} \sum_{n=-N}^{N} [\alpha_{mn}\hat{\varphi} + \beta_{mn}\hat{t}]\, \psi_{mn}.$$

The results of Sections 3 and 5 show that if $\bar{r} \in S$ then each component of $\bar{J}^0$ is in $H(d, d')$. Using formulas (5.6), we have

$$\begin{aligned} \alpha_{mn} &= \frac{f^{1/2}(z_n)}{2\pi} \int_0^{2\pi} \bar{E}^0(\bar{r}) \cdot \hat{\varphi} e^{-im\varphi}\, d\varphi \,|_{z=z_n}, \\ \beta_{mn} &= \frac{f^{1/2}(z_n)}{2\pi} \int_0^{2\pi} \bar{E}^0(\bar{r}) \cdot \hat{t} e^{-im\varphi}\, d\varphi \,|_{z=z_n}. \end{aligned} \tag{6.3}$$

If this is taken together with (3.10), we get

$$\begin{aligned}\alpha_{mn} &= f^{1/2}(z_n)\left[\frac{1}{2}a_1(d_{m+1,n}+d_{m-1,n})+\frac{\cos\theta_0}{2i}a_2(d_{m-1,n}-d_{m+1,n})\right],\\ \beta_{mn} &= f^{1/2}(z_n)\{[a_1 f'(z_n)+a_2\sin\theta_0]\,\alpha(z_n)\,d_{mn}\\ &\quad -\tfrac{1}{2}a_2\cos\theta_0\,\alpha(z_n)\,f'(z_n)[d_{m+1,n}+d_{m-1,n}]\}.\end{aligned}\tag{6.4}$$

7. Derivation of the Galerkin Equations for the Surface Current

Rather than solve (4.11) for $\bar{K}$, it is numerically more convenient to define $\bar{J}$ by

$$\bar{J}(\bar{r}) = v(z)\,\bar{K}(\bar{r}),\tag{7.1}$$

where

$$v(z) = f(z)(1-z^2),$$

and to solve the resulting integral equation for $\bar{J}$. This transformation helps to take into account the unknown[1] singular behavior of $\bar{K}$ at $z=\pm 1$, and enables us to effectively approximate both $\bar{J}$ and its first derivative with respect to z, by the methods of Section 5.

Substitution (7.1) replaces (4.6) by the equation

$$\left\{\frac{1}{2}\eta Z\bar{J}+v\int_S\left[\frac{i\omega\mu}{v}\bar{J}G+\frac{\eta Z}{v}(\hat{n}\times\bar{J})\times\nabla G\right.\right.$$
$$\left.\left.+\frac{1}{i\omega\varepsilon}\left(\nabla\cdot\frac{1}{v}\bar{J}\right)\nabla G\right]dS+\bar{J}^0\right\}_{\tan}=0,\tag{7.2}$$

where

$$\bar{J}^0 = v\bar{E}^0.\tag{7.3}$$

We now make the Galerkin approximation

$$\bar{J}\cong\bar{\mathscr{J}}=\sum_{m=-M}^{M}\sum_{n=-N}^{N}[a_{mn}\hat{\varphi}+b_{mn}\hat{t}]\,\psi_{mn}\tag{7.4}$$

in (7.2), where the ψ_{mn} are defined as in (5.6), and the a_{mn} and b_{mn} are unknown numbers.

[1] If S satisfies Liapunov conditions (see Section 4), then $\bar{K}$ is bounded on S. It is difficult to determine the exact singular behavior of the derivatives of $\bar{K}$ at $z=\pm 1$.

Let us now recall that if $\bar{r}, \bar{r}' \in S$, then

$$R = |\bar{r} - \bar{r}'| = \{f^2 + f^{*2} - 2ff^* \cos(\varphi - \varphi') + (z - z')^2\}^{1/2}, \qquad G = \frac{1}{4\pi} \frac{e^{ik_0 R}}{R}, \tag{7.5}$$

where here and henceforth

$$f \equiv f(z), \qquad f^* \equiv f(z'), \qquad \alpha = \alpha(z), \qquad \alpha^* = \alpha(z'). \tag{7.6}$$

We shall also use the notations

$$\begin{aligned} R^f &= \frac{\partial R}{\partial f} = \frac{f - f^* \cos(\varphi - \varphi')}{R}, \\ R^z &= \frac{\partial R}{\partial z} = \frac{z - z'}{R}, \\ R^\varphi &= \frac{ff^* \sin(\varphi - \varphi')}{R}, \\ (G^f, G^z, G^\varphi) &= (R^f, R^z, R^\varphi) \frac{d}{dR} G. \end{aligned} \tag{7.7}$$

Moreover, writing $\sum_{m,n}$ for $\sum_{m=-M}^{M} \sum_{n=-N}^{N}$, we have

$$\begin{aligned} \hat{n} \times \bar{\mathscr{T}} &= \sum_{m,n} \begin{vmatrix} \hat{\varphi} & \hat{t} & \hat{n} \\ 0 & 0 & 1 \\ a_{mn} & b_{mn} & 0 \end{vmatrix} \psi_{mn} \\ &= \sum_{m,n} [-b_{mn}\hat{\varphi} + a_{mn}\hat{t}]\, \psi_{mn}, \end{aligned} \tag{7.8}$$

and also,

$$\begin{aligned} \nabla G &= \frac{1}{f} G^\varphi \hat{\varphi} + \alpha(f' G^f + G^z)\hat{t} + \alpha(G^f - f' G^z)\hat{n}, \\ \nabla \cdot \left(\frac{1}{v} \bar{\mathscr{T}}\right) &= \frac{1}{f} \sum_{m,n} \left[\frac{a_{mn}}{v} \frac{\partial \psi_{mn}}{\partial \varphi} + \alpha b_{mn} \frac{\partial}{\partial z} \left(\frac{f}{v} \psi_{mn} \right) \right]. \end{aligned} \tag{7.9}$$

Using these results, we get

$$(\hat{n} \times \mathscr{J}) \times \nabla G = \sum_{m,n} \begin{vmatrix} \hat{\varphi} & \hat{t} & \hat{n} \\ -b_{mn} & a_{mn} & 0 \\ (1/f)\, G^{\varphi} & \alpha(f'G^f + G^z) & \alpha(G^f - f'G^z) \end{vmatrix} \psi_{mn}$$

$$= \sum_{m,n} \Big[a_{mn}\alpha(G^f - f'G^z)\hat{\varphi} + b_{mn}\alpha(G^f - f'G^z)\hat{t} \tag{7.10}$$

$$- \left\{ a_{mn} \cdot \frac{1}{f} G^{\varphi} + b_{mn}\alpha(f'G^f + G^z) \right\} \hat{n} \Big] \psi_{mn},$$

as well as

$$\nabla \cdot \left(\frac{1}{\nu} \mathscr{J}\right) \nabla G = \left\{ \frac{1}{f} G^{\varphi}\hat{\varphi} + \alpha(f'G^f + G^z)\hat{t} + \alpha(G^f - f'G^z)\hat{n} \right\}$$

$$\cdot \frac{1}{f} \sum_{m,n} \left[\frac{a_{mn}}{\nu} \frac{\partial \psi_{mn}}{\partial \varphi} + \alpha b_{mn} \frac{\partial}{\partial z}\left(\frac{f}{\nu} \psi_{mn} \right) \right]. \tag{7.11}$$

Substituting (7.4), (7.10), (7.11) as well as the approximation $\mathscr{J}^0$,

$$J^0 \cong \mathscr{J}^0 \equiv \sum_{m,n} [\alpha_{mn}\hat{\varphi} + \beta_{mn}\hat{t}]\, \psi_{mn}, \tag{7.12}$$

into (7.2), and recalling that

$$dS = \frac{f}{\alpha}\, d\varphi\, dz, \tag{7.13}$$

we get

$$\Big\{ \frac{1}{2} \eta Z \sum_{m,n} [a_{mn}\hat{\varphi}^* + b_{mn}\hat{t}^*]\, \psi_{mn}^*$$

$$+ \nu^* \int_{-1}^{1} \int_{0}^{2\pi} \Big\{ \frac{i\omega\mu f}{\alpha\nu} G \sum_{m,n} [a_{mn}\hat{\varphi} + b_{mn}\hat{t}]\, \psi_{mn}$$

$$+ \frac{\eta Z}{\nu} \sum_{m,n} [a_{mn} f(G^f - f'G^z)\hat{\varphi} + b_{mn} f(G^f - f'G^z)\hat{t}$$

$$- \left\{ a_{mn} \frac{1}{\alpha} G^{\varphi} + b_{mn} f(f'G^f + G^z) \right\} \hat{n} \Big\}\, \psi_{mn}$$

$$+ \frac{1}{i\omega\varepsilon} \left\{ \frac{1}{\alpha f} G^{\varphi}\hat{\varphi} + (f'G^f + G^z)\hat{t} + (G^f - f'G^z)\hat{n} \right\}$$

$$\sum_{m,n}\left[\frac{a_{mn}}{v}\frac{\partial\psi_{mn}}{\partial\varphi}+\alpha b_{mn}\frac{\partial}{\partial z'}\left(\frac{f}{v}\psi_{mn}\right)\right]\Bigg\}\,d\varphi\,dz$$

$$+\sum_{m,n}\left[\alpha_{mn}\hat{\varphi}^{*}+\beta_{mn}\hat{t}^{*}\right]\psi_{mn}^{*}\Bigg\}_{\tan}=0, \tag{7.14}$$

where the starred variables denote functions of z' and φ'.

It is convenient to introduce several identities in order to reduce (7.14) to a system of linear algebraic equations. Equations (2.4) yield the identities

$$\begin{aligned}
\hat{\varphi}\cdot\hat{\varphi}^{*}&=\cos(\varphi-\varphi'),\\
\hat{t}\cdot\hat{\varphi}^{*}&=-\alpha(z)\,f'(z)\sin(\varphi'-\varphi),\\
\hat{n}\cdot\hat{\varphi}^{*}&=-\alpha(z)\sin(\varphi'-\varphi),\\
\hat{\varphi}\cdot\hat{t}^{*}&=\alpha(z')\,f'(z')\sin(\varphi'-\varphi),\\
\hat{t}\cdot\hat{t}^{*}&=\alpha(z')\,\alpha(z)[1+f'(z)\,f'(z')\cos(\varphi-\varphi')],\\
\hat{n}\cdot\hat{t}^{*}&=\alpha(z')\,\alpha(z)[f'(z')\cos(\varphi-\varphi')-f'(z)].
\end{aligned} \tag{7.15}$$

These identities are useful for taking components of $\hat{\varphi}^{*}$ and $\hat{t}^{*}$ in (7.14).

Identities (7.16)–(7.23), which follow, serve to achieve further simplicity by enabling us to symbolically eliminate the integrations with respect to φ.

Upon setting

$$G(\bar{r},\bar{r}')=\frac{e^{ik_0|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|}\equiv\sum_{m=-\infty}^{\infty}G_m e^{im(\varphi'-\varphi)} \tag{7.16}$$

it follows that

$$e^{im\varphi'}G_m=\frac{1}{2\pi}\int_0^{2\pi}G(\bar{r},\bar{r}')\,e^{im\varphi}\,d\varphi. \tag{7.17}$$

Furthermore, notice that $G_{-m}=G_m$. Therefore

$$\frac{1}{2\pi}\int_0^{2\pi}G(\bar{r},\bar{r}')\,e^{im\varphi}\cos(\varphi'-\varphi)\,d\varphi=\frac{e^{im\varphi'}}{2}\left[G_{m+1}+G_{m-1}\right] \tag{7.18}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}G(\bar{r},\bar{r}')\,e^{im\varphi}\sin(\varphi-\varphi')\,d\varphi=\frac{e^{im\varphi'}}{2i}\left[G_{m+1}-G_{m-1}\right]. \tag{7.19}$$

By means of integration by parts, we furthermore find that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} e^{im\varphi}\,\frac{\partial G(\bar{r},\bar{r}')}{\partial\varphi}\,d\varphi &= \frac{-im}{2\pi}\int_0^{2\pi} e^{im\varphi}G(\bar{r},\bar{r}')\,d\varphi \\ &= -ime^{im\varphi'}G_m,\end{aligned} \tag{7.20}$$

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} e^{im\varphi}\cos(\varphi-\varphi')\,\frac{\partial G}{\partial\varphi}\,d\varphi \\ &\quad = -\frac{ie^{im\varphi'}}{2}\,[(m+1)\,G_{m+1}+(m-1)\,G_{m-1}],\end{aligned} \tag{7.21}$$

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} e^{im\varphi}\sin(\varphi-\varphi')\,\frac{\partial G(\bar{r},\bar{r}')}{\partial\varphi}\,d\varphi \\ &\quad = -\frac{e^{im\varphi'}}{2}\,[(m+1)\,G_{m+1}-(m-1)\,G_{m-1}].\end{aligned} \tag{7.22}$$

We use (7.15) in(7.14) to take components of $\hat{\varphi}$ and $\hat{t}$. Then, recalling that

$$\psi_{mn}(z,\varphi)=e^{im\varphi}\theta_n(z), \tag{7.23}$$

where

$$\theta_n(z)=\frac{\nu(z)}{f(z)^{1/2}}\,\mathrm{sinc}\left[\frac{\omega(z)-nh}{h}\right], \tag{7.24}$$

where $\nu(z)$ is given in (5.3), and where

$$\omega(z)=\log\left(\frac{1+z}{1-z}\right), \tag{7.25}$$

we can symbolically carry out the integrations with respect to φ' in the equation resulting from (7.14), by using (7.17)–(7.22). Upon equating coefficients of $e^{im\varphi}$ in the resulting equations, we obtain

$$\begin{aligned}&\frac{1}{2}\eta Z\sum_n a_{mn}\theta_n+\nu\sum_{n=-N}^{N}\,[P^{mn}a_{mn}+Q^{mn}b_{mn}]=-\sum_{n=-N}^{N}\alpha_{mn}\theta_n, \\ &\frac{1}{2}\eta Z\sum_n b_{mn}\theta_n+\nu\sum_{n=-N}^{N}\,[R^{mn}a_{mn}+S^{mn}b_{mn}]=-\sum_{n=-N}^{N}\beta_{mn}\theta_n, \\ &\qquad m=-M,\quad -M+1,\ldots,M.\end{aligned} \tag{7.26}$$

The coefficients P^{mn}, Q^{mn}, R^{mn} and S^{mn} depend on z', and are given by

$$\begin{aligned} P^{mn} = \pi \int_{-1}^{1} \frac{\theta_n}{v} \Bigg\{ & G_{m+1}\left[\frac{i\omega\mu f}{\alpha} + (m+1)\,\eta Z + \frac{m(m+1)}{i\omega\varepsilon\alpha f}\right] \\ & + G_{m-1}\left[\frac{i\omega\mu f}{\alpha} - (m-1)\,\eta Z + \frac{m(m-1)}{i\omega\varepsilon\alpha f}\right] \\ & + G^f_{m+1}\left[\eta Z f + \frac{m}{i\omega\varepsilon\alpha}\right] \\ & + G^f_{m-1}\left[\eta Z f - \frac{m}{i\omega\varepsilon\alpha}\right] \\ & - (G^z_{m+1} + G^z_{m-1})\,\eta Z f f' \Bigg\}\, dz, \end{aligned} \tag{7.27}$$

$$\begin{aligned} Q^{mn} = \pi \int_{-1}^{1} \Bigg[& \frac{\theta_n}{v} \left\{ (G_{m+1} - G_{m-1})\,\omega\mu f f' - (G^z_{m+1} - G^z_{m-1})\,\frac{\eta Z f}{i\alpha} \right\} \\ & + \left(\frac{f\theta_n}{v}\right)' \Bigg\{ -\frac{1}{\omega\varepsilon f}\left[(m+1)\,G_{m+1} + (m-1)\,G_{m-1}\right] \\ & - \frac{1}{\omega\varepsilon}(G^f_{m+1} - G^f_{m-1}) \Bigg\} \Bigg]\, dz, \end{aligned} \tag{7.28}$$

$$\begin{aligned} R^{mn} = \pi \int_{-1}^{1} \frac{\theta_n}{v} \Bigg\{ & G_{m+1}\alpha^* f^{*\prime} \left[\frac{-\omega\mu f}{\alpha} + i(m+1)\,\eta Z + \frac{m(m+1)}{\omega\varepsilon\alpha f}\right] \\ & + G_{m-1}\alpha^* f^{*\prime} \left[\frac{\omega\mu f}{\alpha} + i(m-1)\,\eta Z - \frac{m(m-1)}{\omega\varepsilon\alpha f}\right] \\ & + G_m \alpha^* [-2imf'\eta Z] \\ & + G^f_{m+1}\alpha^* f^{*\prime} \left[\frac{-\eta Z f}{i} + \frac{m}{\omega\varepsilon\alpha}\right] \\ & + G^f_{m-1}\,\alpha^* f^{*\prime} \left[\frac{\eta Z f}{i} + \frac{m}{\omega\varepsilon\alpha}\right] \\ & + \alpha^* f^{*\prime}\,\frac{\eta Z f f'}{i}\,[G^z_{m+1} - G^z_{m-1}] \\ & + \frac{2m\alpha^*}{\omega\varepsilon\alpha} G^z_m \Bigg\}\, dz, \end{aligned} \tag{7.29}$$

$$
\begin{aligned}
S^{mn} = \pi \int_{-1}^{1} \Bigg[\left(\frac{\theta_n}{\nu}\right) \Bigg\{ & \alpha^* f^{*\prime} i\omega\mu f f' [G_{m+1} + G_{m-1}] + 2\alpha^* i\omega\mu f G_m \\
& + 2\eta \frac{Z f \alpha^*}{\alpha} G_m^f - \eta \frac{Z f \alpha^* f^{*\prime}}{\alpha} [G_{m+1}^z + G_{m-1}^z] \Bigg\} \\
& + \left(\frac{f\theta_n}{\nu}\right)' \Bigg\{ \frac{\alpha^* f^{*\prime}}{i\omega\varepsilon f} [(m+1) G_{m+1} - (m-1) G_{m-1}] \\
& + \frac{\alpha^* f^{*\prime}}{i\omega\varepsilon} [G_{m+1}^f + G_{m-1}^f] \\
& + \frac{2\alpha^*}{i\omega\varepsilon} G_m^z \Bigg\} \Bigg] \, dz.
\end{aligned} \tag{7.30}
$$

In these equations $f^{*\prime} = f'(z')$, $\alpha^* = [1 + (f^{*\prime})^2]^{-1/2}$. Next, setting $z' = z_l = \tanh(lh/2)$ in (7.26) and using the relations

$$
\begin{aligned}
\theta_n(z_l) &= 0 && \text{if} \quad n \neq l \\
&= \nu(z_l)/f^{1/2}(z_l) && \text{if} \quad n = l
\end{aligned} \tag{7.31}
$$

we arrive at the system

$$
\begin{aligned}
\frac{1}{2} \eta Z a_{ml} + f^{1/2}(z_l) \sum_{n=-N}^{N} [P_l^{mn} a_{mn} + Q_l^{mn} b_{mn}] &= -\alpha_{ml}, \\
\frac{1}{2} \eta Z b_{ml} + f^{1/2}(z_l) \sum_{n=-N}^{N} [R_l^{mn} a_{mn} + S_l^{mn} b_{mn}] &= -\beta_{ml},
\end{aligned} \tag{7.32}
$$

where we have used the notation

$$
\begin{aligned}
P_l^{mn} &= P^{mn}(z_l), & Q_l^{mn} &= Q^{mn}(z_l), \\
R_l^{mn} &= R^{mn}(z_l), & S_l^{mn} &= S^{mn}(z_l).
\end{aligned} \tag{7.33}
$$

System (7.32) is a block diagonal system of the form

$$
\begin{bmatrix} B_{-M} & & & \\ & B_{-M+1} & & \\ & & \ddots & \\ & & & B_M \end{bmatrix}
\begin{bmatrix} \bar{a}_{-M} \\ \bar{a}_{-M+1} \\ \vdots \\ \bar{a}_M \end{bmatrix}
=
\begin{bmatrix} \bar{\alpha}_{-M} \\ \bar{\alpha}_{-M+1} \\ \vdots \\ \bar{\alpha}_M \end{bmatrix}, \tag{7.34}
$$

where each B_m is a complex matrix of order $2(2N+1)$, and (since $G_{-m} = G_m$) $B_m = B_{-m}$. The mth system

$$
B_m \bar{a}_m = \bar{\alpha}_m \tag{7.35}
$$

in (7.34) corresponds to all of the equations (7.32), for fixed m. Thus if we denote by A_l^{mn} the 2×2 matrix

$$A_l^{mn} = f^{1/2}(z_l) \begin{bmatrix} P_l^{mn} & Q_l^{mn} \\ R_l^{mn} & S_l^{mn} \end{bmatrix} + \tfrac{1}{2}\eta Z \delta_{nl} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.36}$$

then

$$B_m = \begin{bmatrix} A_{-N}^{m,-N} & A_{-N}^{m,-N+1} & \cdots & A_{-N}^{mN} \\ A_{-N+1}^{m,-N} & A_{-N+1}^{m,-N+1} & \cdots & A_{-N+1}^{mN} \\ & \cdots & & \\ A_N^{m,-N} & A_N^{m,-N+1} & \cdots & A_N^{mN} \end{bmatrix} \tag{7.37}$$

and

$$\bar{a}_m = \begin{bmatrix} a_{m,-N} \\ b_{m,-N} \\ a_{m,-N+1} \\ b_{m,-N+1} \\ \vdots \\ a_{mN} \\ b_{mN} \end{bmatrix} \qquad \bar{a}_m = -\begin{bmatrix} \alpha_{m,-N} \\ \beta_{m,-N} \\ \alpha_{m,-N+1} \\ \beta_{m,-N+1} \\ \vdots \\ \alpha_{mN} \\ \beta_{mN} \end{bmatrix}. \tag{7.38}$$

7(b) EVALUATION OF P^{mn}, Q^{mn}, R^{mn} AND S^{mn}

It is convenient to set

$$\theta_n/v = \gamma_n, \qquad \left(\frac{\theta_n f}{v}\right)' = \delta_n. \tag{7.39}$$

The following integrals appear in (7.27)–(7.30).

$$I_{mn}^{(1)} = \int_{-1}^{1} \gamma_n (f/\alpha)\, G_m \, dz, \qquad I_{mn}^{(4)} = \int_{-1}^{1} \gamma_n f f' G_m \, dz,$$

$$I_{mn}^{(2)} = \int_{-1}^{1} \gamma_n G_m \, dz \ (m \neq 0), \qquad I_{mn}^{(5)} = \int_{-1}^{1} \gamma_n f' G_m \, dz \ (m \neq 0),$$

$$I_{mn}^{(3)} = \int_{-1}^{1} [\gamma_n/(\alpha f)]\, G_m \, dz \ (m \neq 0), \qquad I_{mn}^{(6)} = \int_{-1}^{1} \gamma_n f G_m \, dz,$$

$$J_{mn}^{(1)} = \int_{-1}^{1} \gamma_n f G_m^f \, dz, \qquad K_{mn}^{(1)} = \int_{-1}^{1} \gamma_n f f' G_m^z \, dz,$$

$$J_{mn}^{(2)} = \int_{-1}^{1} \gamma_n (1/\alpha)\, G_m^f \, dz, \qquad K_{mn}^{(2)} = \int_{-1}^{1} \gamma_n (1/\alpha)\, G_m^z \, dz \ (m \neq 0),$$

$$J_{mn}^{(3)} = \int_{-1}^{1} \gamma_n(f/\alpha)\, G_m^f\, dz, \qquad K_{mn}^{(3)} = \int_{-1}^{1} \gamma_n(f/\alpha)\, G_m^z\, dz,$$

$$L_{mn}^{(1)} = \int_{-1}^{1} \delta_n(1/f)\, G_m\, dz\ (m \neq 0),$$

$$M_{mn}^{(1)} = \int_{-1}^{1} \delta_n G_m^f\, dz,$$

$$N_{mn}^{(1)} = \int_{-1}^{1} \delta_n G_m^z\, dz. \tag{7.40}$$

Notations (7.40) enable us to express (7.29), (7.30) in the "more computable" form

$$\begin{aligned} P^{mn} = \pi\{ & i\omega\mu(I_{m+1,n}^{(1)} + I_{m-1,n}^{(1)}) + \eta Z[(m+1)\, I_{m+1,n}^{(2)} - (m-1)\, I_{m-1,n}^{(2)}] \\ & + \frac{m}{i\omega\varepsilon}[(m+1)\, I_{m+1,n}^{(3)} + (m-1)\, I_{m-1,n}^{(3)}] \\ & + \eta Z(J_{m+1,n}^{(1)} + J_{m-1,n}^{(1)} - K_{m+1,n}^{(1)} - K_{m-1,n}^{(1)}) \\ & + \frac{m}{i\omega e}(J_{m+1,n}^{(2)} - J_{m-1,n}^{(2)})\Big\}, \end{aligned} \tag{7.41}$$

$$\begin{aligned} Q^{mn} = \pi\Big\{ & \omega\mu(I_{m+1,n}^{(4)} - I_{m-1,n}^{(4)}) - \frac{\eta Z}{i}(K_{m+1,n}^{(3)} - K_{m-1,n}^{(3)}) \\ & - \frac{1}{\omega\varepsilon}[(m+1)\, L_{m+1,n}^{(1)} + (m-1)\, L_{m-1,n}^{(1)} + M_{m+1,n}^{(1)} - M_{m-1,n}^{(1)}]\Big\}, \end{aligned} \tag{7.42}$$

$$\begin{aligned} R^{mn} = \pi\alpha^* f^{*\prime}\Big\{ & -\omega\mu(I_{m+1,n}^{(1)} - I_{m-1,n}^{(1)}) \\ & + i\eta Z[(m+1)\, I_{m+1,n}^{(2)} + (m-1)\, I_{m+1,n}^{(2)}] \\ & + \frac{m}{\omega\varepsilon}[(m+1)\, I_{m+1,n}^{(3)} - (m-1)\, I_{m-1,n}^{(3)}] - \frac{2i\eta Zm}{f^{*\prime}} I_{mn}^{(5)} \\ & - \frac{\eta Z}{i}(J_{m+1,n}^{(1)} - J_{m-1,n}^{(1)} - K_{m+1,n}^{(1)} + K_{m-1,n}^{(1)}) \\ & + \frac{m}{\omega\varepsilon}\Big(J_{m+1,n}^{(2)} + J_{m-1,n}^{(2)} + \frac{2}{f^{*\prime}} K_{mn}^{(2)}\Big)\Big\} \end{aligned} \tag{7.43}$$

and

$$
\begin{aligned}
S^{mn} = \pi\alpha^* f^{*\prime} \Bigg\{ & i\omega\mu\left(I^{(4)}_{m+1,n} + I^{(4)}_{m-1,n} + \frac{2}{f^{*\prime}} I^{(6)}_{mn}\right) \\
& - Z\left(K^{(3)}_{m+1,n} + K^{(3)}_{m-1,n} - \frac{2}{f^{*\prime}} J^{(3)}_{mn}\right) \\
& + \frac{1}{i\omega\varepsilon}\Bigg[(m+1)\, L^{(1)}_{m+1,n} - (m-1)\, L^{(1)}_{m-1,n} \\
& + M^{(1)}_{m+1,n} + M^{(1)}_{m-1,n} + \frac{2}{f^{*\prime}} N^{(1)}_{mn}\Bigg]\Bigg\}.
\end{aligned}
\tag{7.44}
$$

Integrals (7.40) can be simultaneously evaluated for all m, n by evaluating the quantities f, f', $\alpha = (1 + f'^2)^{-1/2}$, $f^{*\prime}$ and $\alpha^* = (1 + f^{*\prime 2})^{-1/2}$, θ_n ($n = -N, -N+1, \ldots, N$), G_m, G^f_m, and G^f_m. In view of (A.2), we set

$$
a(z') = \frac{1}{4\pi^2} \frac{1}{f(z')} \frac{1}{[1 + f'(z')^2]} \log\left(\frac{1+z'}{1-z'}\right), \tag{7.45}
$$

$$
\mu_s = \frac{z' e^{sh} - 1}{e^{sh} + 1}, \qquad \nu_s = \frac{z' e^{sh} + 1}{e^{sh} + 1}. \tag{7.46}
$$

Then by splitting the range of integration into an integral with respect to z from -1 to z' plus an integral with respect to z from z' to 1, we use the formulas

$$
\begin{aligned}
& \int_{-1}^{1} G_m(z, z')\, g(z)\, dz \\
& \quad \cong h \sum_{s=-N}^{N} \frac{e^{sh}}{(1 + e^{sh})^2} \{(1 + z')\, G_m(\mu_s) + (1 - z')\, G_m(\nu_s, z')\, g(\nu_s)\},
\end{aligned}
\tag{7.47}
$$

$$
\begin{aligned}
& \int_{-1}^{1} G^f_m(z, z')\, g(z)\, dz \\
& \quad \cong h \sum_{s=-N}^{N} \frac{e^{sh}}{(1 + e^{sh})^2} \{(1 + z')\, G^f_m(\mu_s, z')\, g(\mu_s) + (1 - z')\, G^f_m(\nu_s, z')\, g(\nu_s)\} \\
& \quad\quad + f'(z')\, a(z')\, g(z'),
\end{aligned}
\tag{7.48}
$$

$$\begin{aligned}
&\int_{-1}^{1} G_m^z(z, z')\, g(z)\, dz \\
&\quad \cong h \sum_{s=-N}^{N} \frac{e^{sh}}{(1+e^{sh})^2} \\
&\qquad \times \{(1+z')\, G_m^z(\mu_s, z')\, g(\mu_s) + (1-z')\, G_m^z(\nu_s, z')\, g(\nu_s)\} \\
&\qquad + a(z')\, g(z'). \qquad (7.49)
\end{aligned}$$

to approximate the various integrals (7.40).

In (7.47), (7.48) and (7.49), g is a suitable analytic factor of G_m, G_m^f or G_m^z, as dictated by (7.40). The additional terms in (7.48) and (7.49) arise as a consequence of the identity

$$P.V. \int_{-1}^{1} \frac{dz}{z - z'} = \log \frac{1+z'}{1-z'} \qquad (7.50)$$

whereas, if the formula used in (7.47) is used to approximate this singular integral, one has the identity

$$h \sum_{s=-N}^{N} \frac{e^{sh}}{(1+e^{sh})^2} \left\{ \frac{1+z'}{\mu_s - z'} + \frac{1-z'}{\nu_s - z'} \right\} = 0. \qquad (7.51)$$

That is, whereas formula (7.47) is accurate for evaluating

$$\int_{-1}^{1} g(z)\, dz, \qquad \int_{-1}^{1} g(z) \log |z - z'|\, dz, \qquad (7.52)$$

where g is analytic on $(-1, 1)$, when applied to the approximation of

$$\begin{aligned}
&I = P.V. \int_{-1}^{1} \frac{g(z)}{z - z'}\, dz \\
&\left(= \int_{-1}^{1} \frac{g(z) - g(z')}{z - z'}\, dz + g(z')\, P.V. \int_{-1}^{1} \frac{dz}{z - z'} \right) \qquad (7.53)
\end{aligned}$$

it yields an accurate approximation to

$$\int_{-1}^{1} \frac{g(z) - g(z')}{z - z'}\, dz. \qquad (7.54)$$

8. Approximation of the Scattered Field

The scattered field $\bar{E}^s$ is expressed in terms of $\bar{K}$ by means of integral (4.7). Once $\bar{\mathscr{T}}$ has been obtained by means of Section 7, we form an approximation $\bar{\mathscr{K}}$ of $\bar{K}$ by means of the equation (see Eq. (7.1))

$$\bar{\mathscr{K}} = \bar{\mathscr{K}}(z, \varphi) = (1/v(z))\, \bar{\mathscr{T}}(z, \varphi), \tag{8.1}$$

and we substitute $\bar{\mathscr{K}}$ for $\bar{K}$ in (4.7) to get an expression for an approximation $\bar{\mathscr{E}}^s$ of $\bar{E}^s$. In this section we shall give a detailed description of the evaluation of $\bar{\mathscr{E}}^s$.

We shall approximate $\bar{\mathscr{E}}^s(\bar{r})$ for

$$\bar{r}' = \rho \cos \varphi' \hat{x}' + \rho \sin \varphi' \hat{y}' + z' \hat{z}', \tag{8.2}$$

where $\rho > f(z')$. If ρ is close to $f(z)$, i.e., if $\bar{r}'$ is close to the surface (say, $|\bar{r}' - S| \leqslant 0.2$ if $N = 10$) then we recommend that the integration methods of Section 7 be used to evaluate the integrals. This would involve splitting the integrals from -1 to 1 into integrals from -1 to z' and from z' to 1, as in (7.47). For the sake of simplicity, we shall describe an algorithm for evaluating $\bar{\mathscr{E}}^s$, which is valid if $\bar{r}'$ is not arbitrarily close to S (say, $|\bar{r}' - S| > 0.2$ if $N \geqslant 10$). In this latter case it is convenient to integrate by parts in the integrals with respect to z which involve θ_n', so that the resulting integrals involve θ_n. This latter procedure enables us to avoid numerical integration, by means of the approximation

$$\int_{-1}^{1} H(z)\, \omega_n(z)\, dz \cong h \frac{H(z_n)}{f^{1/2}(z_n)\, \omega'(z_n)}, \tag{8.3}$$

which we know to be accurate, by Theorem 5.2, where $\omega_n = \theta_n / v$. See Fig. 8.1

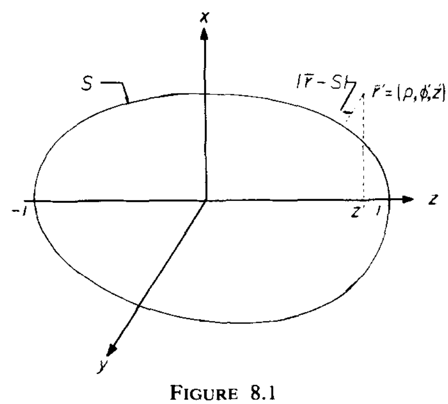

FIGURE 8.1

Using (4.7), the scattered field may be expressed in the form

$$\bar{E}^s(\bar{r}) = E_x^s \hat{x} + E_y^s \hat{y} + E_x^s \hat{z}, \tag{8.4}$$

where E_x^s, E_y^s and E_z^s are scalar quantities. Upon substituting approximation (7.4) into (4.7) we get

$$E_x^s \cong \sum_{m,n} [a_{mn} P_{mn} + b_{mn} Q_{mn}], \tag{8.5}$$

$$E_y^s \cong \sum_{m,n} [a_{mn} R_{mn} + b_{mn} S_{mn}], \tag{8.6}$$

$$E_z^s \cong \sum_{m,n} [a_{mn} T_{mn} + b_{mn} U_{mn}]. \tag{8.7}$$

Relations (7.7)–(7.11) and (7.15)–(7.22) enable us to obtain explicit expression for P^{mn},..., U^{mn}. Setting

$$\omega_n = \omega_n(z) = \frac{\theta_n(z)}{v(z)}, \tag{8.8}$$

we get

$$\begin{aligned}
\mathscr{E}^s = \int_{-1}^{1} \int_{0}^{2\pi} \Bigg\{ & \frac{i\omega\mu}{\alpha} fG \sum_{m,n} [a_{mn}(-\sin\varphi\hat{x} + \cos\varphi\hat{y}) \\
& + b_{mn}(f'\alpha \cos\varphi\hat{x} + f'\alpha \sin\varphi\hat{y} + \alpha\hat{z})]\, e^{im\varphi}\omega_n \\
& + \eta Z \sum_{m,n} \Bigg[a_{mn} f(G^f - f'G^z)(-\sin\varphi\hat{x} + \cos\varphi\hat{y}) \\
& + b_{mn} f(G^f - f'G^z)(f'\alpha \cos\varphi\hat{x} + f'\alpha \sin\varphi\hat{y} + \alpha\hat{z}) \\
& - \Bigg\{ a_{mn} \frac{1}{\alpha} G^\varphi(\alpha \cos\varphi\hat{x} + \alpha \sin\varphi\hat{y} - f'\alpha\hat{z}) \\
& + b_{mn} f(f'G^f + G^z)(\alpha \cos\varphi\hat{x} + \alpha \sin\varphi\hat{y} - f'\alpha\hat{z}) \Bigg\} \Bigg]\, e^{im\varphi}\omega_n \\
& + \frac{1}{i\omega\varepsilon} \Bigg\{ \frac{1}{\alpha f} G^\varphi(-\sin\varphi\hat{x} + \cos\varphi\hat{y}) \\
& + (f'G^f + G^z)(f'\alpha \cos\varphi\hat{x} + f'\alpha \sin\varphi\hat{y} + \alpha\hat{z}) \\
& + (G^f - f'G^z)(\alpha \cos\varphi\hat{x} + \alpha \sin\varphi\hat{y} - f'\alpha\hat{z}) \Bigg\} \\
& \cdot \sum_{m,n} [a_{mn} im\omega_n + \alpha b_{mn}(f\omega_n)']\, e^{im\varphi} \Bigg\}\, d\varphi\, dz.
\end{aligned} \tag{8.9}$$

Here $G = e^{ik_0R}/4\pi R$; $R = \{(z-z')^2 + f^2 + \rho^2 - 2f'\rho\cos(\varphi-\varphi')\}^{1/2}$. Collecting coefficients as indicated in (8.5)–(8.7), we get

$$\begin{aligned}
P_{mn} = \pi e^{im\phi'} \int_{-1}^{1} \Bigg\{ & \frac{-\omega\mu f}{\alpha} [e^{i\phi'} G_{m+1} - e^{-i\phi'} G_{m-1}] \\
& + i\eta Z [f\{(G^f_{m+1} - f' G^z_{m+1})\, e^{i\phi'} - (G^f_{m-1} - f' G^z_{m-1})\, e^{-i\phi'}\} \\
& + (m+1)\, e^{i\phi'} G_{m+1} + (m-1)\, e^{-i\phi'} G_{m-1}] \\
& + \frac{m}{\omega\varepsilon\alpha} \bigg[\frac{1}{f} \{(m+1)\, e^{i\phi} G_{m+1} - (m-1)\, e^{-i\phi} G_{m-1}\} \\
& + \{e^{i\phi} G^f_{m+1} + e^{-i\phi} G^f_{m-1}\}\bigg]\Bigg\}\, \omega_n\, dz;
\end{aligned} \tag{8.10}$$

$$\begin{aligned}
Q_{mn} = \pi e^{im\phi'} \int_{-1}^{1} \Bigg\{ & \bigg[i\omega\mu f f' [e^{i\phi'} G_{m+1} + e^{-i\phi'} G_{m-1}] \\
& - \frac{\eta Z f}{\alpha} [e^{i\phi'} G^z_{m+1} + e^{-i\phi'} G^z_{m-1}]\bigg]\, \omega_n \\
& + \frac{1}{i\omega\varepsilon} \bigg[\frac{1}{f} ((m+1)\, e^{i\phi'} G_{m+1} - (m-1)\, e^{-i\phi'} G_{m-1}) \\
& + e^{i\phi'} G^f_{m+1} + e^{-i\phi'} G^f_{m-1}\bigg] (f\omega_n)' \Bigg\}\, dz;
\end{aligned} \tag{8.11}$$

$$\begin{aligned}
R_{mn} = \pi e^{im\phi'} \int_{-1}^{1} \Bigg\{ & \frac{i\omega\mu f}{\alpha} (e^{i\phi'} G_{m+1} + e^{-i\phi'} G_{m-1}) \\
& + \eta Z [f(e^{i\phi'}(G^f_{m+1} - f' G^z_{m+1}) + e^{-i\phi'}(G^f_{m-1} - f' G^z_{m-1})) \\
& + (m+1)\, e^{i\phi'} G_{m+1} - (m-1)\, e^{-i\phi'} G_{m-1}] \\
& - \frac{im}{\alpha\omega\varepsilon} \bigg[\frac{1}{f} ((m+1)\, e^{i\phi'} G_{m+1} + (m-1)\, e^{-i\phi'} G_{m-1}) \\
& + e^{i\phi'} G^f_{m+1} - e^{-i\phi'} G^f_{m-1}\bigg]\Bigg\}\, \omega_n\, dz;
\end{aligned} \tag{8.12}$$

$$\begin{aligned}
S_{mn} = \pi e^{im\phi'} \int_{-1}^{1} \Bigg\{ & \bigg[\omega\mu f f' (e^{i\phi'} G_{m+1} - e^{-i\phi'} G_{m-1}) \\
& + \frac{i\eta Z f}{\alpha} [e^{i\phi'} G^z_{m+1} - e^{-i\phi'} G^z_{m-1}]\bigg]\, \omega_n \\
& - \frac{1}{\omega\varepsilon} \bigg[\frac{1}{f} ((m+1)\, e^{i\phi'} G_{m+1} + (m-1)\, e^{-i\phi'} G_{m-1}) \\
& + e^{i\phi'} G^f_{m+1} - e^{-i\phi'} G^f_{m-1}\bigg] (f\omega_n)' \Bigg\}\, dz;
\end{aligned} \tag{8.13}$$

$$T_{mn} = -2\pi m e^{im\phi'} \int_{-1}^{1} \left\{ i\eta Z f' G_m - \frac{1}{\omega\varepsilon\alpha} G_m^z \right\} \omega_n \, dz; \tag{8.14}$$

$$U_{mn} = 2\pi e^{im\phi'} \int_{-1}^{1} \left\{ \left[i\omega\mu f G_m + \frac{\eta Z f}{\alpha} G_m^f \right] \omega_n + \frac{1}{i\omega\varepsilon} G_m^z (f\omega_n)' \right\} dz. \tag{8.15}$$

Let us next eliminate the $(f\omega_n)'$ terms which appear in Q_{mn}, S_{mn} and U_{mn} above. Setting

$$\mathscr{T}_{mn} = \int_{-1}^{1} \left(\frac{1}{f} G_m \right) (f\omega_n)' \, dz \tag{8.16}$$

we have, upon integration by parts,

$$\mathscr{T}_{mn} = \frac{1}{f} G_m f\omega_n \bigg]_{-1}^{1} - \int_{-1}^{1} \left(\frac{1}{f} G_m \right)' f\omega_n dz. \tag{8.17}$$

Under the assumption made on S in Section 1, $\bar{K}$ is bounded on S, and therefore it follows, upon replacing ω_n by $\bar{K}$ in (8.15), that the first term on the right-hand side of (8.15) vanishes, provided that $m \neq 0$ (see (A.1)). However, inspection of Q_{mn} and S_{mn} shows that we need never evaluate $\mathscr{T}_{mn}$ if $m = 0$. Thus

$$\mathscr{T}_{mn} = \int_{-1}^{1} \frac{f'}{f} G_m \omega_n \, dz - \int_{-1}^{1} G_m' \omega_n \, dz, \qquad m \geqslant 1, \tag{8.18}$$

where $G_m' = dG_m/dz$.

Similarly, we have, for all $m \geqslant 0$,

$$K_{mn} \equiv \int_{-1}^{1} G_m^f (f\omega_n)' \, dz = -\int_{-1}^{1} (G_m^f)' f\omega_n \, dz \tag{8.19}$$

and

$$L_{mn} \equiv \int_{-1}^{1} G_m^z (f\omega_n)' \, dz = -\int_{-1}^{1} (G_m^z)' f\omega_n \, dz, \tag{8.20}$$

these being the only remaining terms requiring integration by parts in (8.10)–(8.15).

Hence, we make the definitions

$$\begin{aligned}
I^1_{mn} &= \int_{-1}^{1} \frac{f}{\alpha} G_m \omega_n \, dz, \ m \geqslant 0; & J^2_{mn} &= \int_{-1}^{1} \frac{1}{\alpha} G^f_m \omega_n \, dz, \ m \geqslant 0; \\
I^2_{mn} &= \int_{-1}^{1} G_m \omega_n \, dz, \ m \geqslant 1; & J^3_{mn} &= \int_{-1}^{1} \frac{f}{\alpha} G^f_m \omega_n \, dz, \quad m \geqslant 0; \\
I^3_{mn} &= \int_{-1}^{1} \frac{1}{f\alpha} G_m \omega_n \, dz, \ m \geqslant 1; & K^1_{mn} &= \int_{-1}^{1} ff' G^z_m \omega_n \, dz, \ m \geqslant 0; \\
I^4_{mn} &= \int_{-1}^{1} ff' G_m \omega_n \, dz, \ m \geqslant 0; & K^2_{mn} &= \int_{-1}^{1} \frac{f}{\alpha} G^z_m \omega_n \, dz, \ m \geqslant 0; \\
I^5_{mn} &= \int_{-1}^{1} f' G_m \omega_n \, dz, \ m \geqslant 1; & K^3_{mn} &= \int_{-1}^{1} \frac{1}{\alpha} G^z_m \omega_n \, dz, \ m \geqslant 1; \\
I^6_{mn} &= \int_{-1}^{1} f G_m \omega_n \, dz, \ m \geqslant 0; & IP_{mn} &= \int_{-1}^{1} G'_m \omega_n \, dz, \ m \geqslant 1; \\
I^7_{mn} &= \int_{-1}^{1} \frac{f'}{f} G_m \omega_n \, dz, \ m \geqslant 1; & JP_{mn} &= \int_{-1}^{1} (G^f_m)' f \omega_n \, dz, \ m \geqslant 0; \\
J^1_{mn} &= \int_{-1}^{1} f G^f_m \omega_n \, dz, \ m \geqslant 0; & KP_{mn} &= \int_{-1}^{1} (G^z_m)' f \omega_n \, dz, \ m \geqslant 0.
\end{aligned} \tag{8.21}$$

Some of the quantities in (8.21) are not required for $m = 0$, since their coefficient in (8.10)–(8.15) is zero. In some cases this is fortunate, since some of the above integrals do not exist when m is taken to be zero. In terms of the above integrals (8.21) we may express terms (8.10)–(8.15) as follows.

$$\begin{aligned}
P_{mn} = \pi e^{im\varphi} \Big\{ &-\omega\mu [e^{i\varphi} I^1_{m+1,n} - e^{-i\varphi} I^1_{m-1,n}] \\
&+ i\eta Z [e^{i\varphi}(J^1_{m+1,n} - K^1_{m+1,n}) - e^{-i\varphi}(J^1_{m-1,n} - K^1_{m-1,n}) \\
&+ (m+1)\, e^{i\varphi} I^2_{m+1,n} + (m-1)\, e^{-i\varphi} I^2_{m-1,n}] \\
&+ \frac{m}{\omega\varepsilon} [(m+1)\, e^{i\varphi} I^3_{m+1,n} - (m-1)\, e^{i\varphi} I^3_{m-1,n} \\
&+ e^{i\varphi} J^2_{m+1,n} + e^{-i\varphi} J^2_{m-1,n}] \Big\};
\end{aligned} \tag{8.22}$$

$$
\begin{aligned}
Q_{mn} = \pi e^{im\varphi} \Big\{ & i\omega\mu [e^{i\varphi} I^4_{m+1,n} + e^{-i\varphi} I^4_{m-1,n}] \\
& - \eta Z [e^{i\varphi} K^2_{m+1,n} + e^{-i\varphi} K^2_{m-1,n}] \\
& + \frac{1}{i\omega\varepsilon} [(m+1)\, e^{i\varphi} (I^7_{m+1,n} - IP_{m+1,n}) \\
& - (m-1)\, e^{-i\varphi} (I^7_{m+1,n} - IP_{m-1,n}) \\
& - e^{i\varphi} JP_{m+1,n} - e^{-i\varphi} JP_{m-1,n}] \Big\} ;
\end{aligned} \tag{8.23}
$$

$$
\begin{aligned}
R_{mn} = \pi e^{im\varphi} \Big\{ & i\omega\mu [e^{i\varphi} I^1_{m+1,n} + e^{-i\varphi} I^1_{m-1,n}] \\
& + \eta Z [e^{i\varphi} (J^1_{m+1,n} - K^1_{m+1,n}) + e^{-i\varphi} (J^1_{m-1,n} - K^1_{m-1,n}) \\
& + (m+1)\, e^{i\varphi} I^2_{m+1,n} - (m-1)\, e^{-i\varphi} I^2_{m-1,n}] \\
& - \frac{im}{\omega\varepsilon} [(m+1)\, e^{i\varphi} I^3_{m+1,n} + (m-1)\, e^{-i\varphi} I^3_{m-1,n} \\
& + e^{i\varphi} J^2_{m+1,n} - e^{-i\varphi} J^2_{m-1,n}] \Big\} ;
\end{aligned} \tag{8.24}
$$

$$
\begin{aligned}
S_{mn} = \pi e^{im\varphi} \Big\{ & \omega\mu [e^{i\varphi} I^4_{m+1,n} - e^{-i\varphi} I^4_{m-1,n}] \\
& + i\eta Z [e^{i\varphi} K^2_{m+1,n} - e^{-i\varphi} K^2_{m-1,n}] \\
& - \frac{1}{\omega\varepsilon} [(m+1)\, e^{i\varphi} (I^7_{m+1,n} - IP_{m+1,n}) \\
& + (m-1)\, e^{-i\varphi} (I^7_{m-1,n} - IP_{m-1,n}) \\
& - e^{i\varphi} JP_{m+1,n} + e^{-i\varphi} JP_{m-1,n}] \Big\} ;
\end{aligned} \tag{8.25}
$$

$$
T_{mn} = -2\pi m e^{im\varphi} \Big\{ i\eta Z I^5_{mn} - \frac{1}{\omega\varepsilon} K^3_{mn} \Big\} ; \tag{8.26}
$$

$$
U_{mn} = 2\pi e^{im\varphi} \Big\{ i\omega\mu I^6_{mn} + \eta Z J^3_{mn} - \frac{1}{i\omega\varepsilon} KP_{mn} \Big\} . \tag{8.27}
$$

Each of the integrals (8.21) may now be accurately evaluated using the one-term formula (8.3), provided that $\bar{r}'$ is not unduly close to S. We shall assume this to be the case. We shall, however, require G_m, G^f_m, G^z_m, dG_m/dz, dG^f_m/dz and dG^z_m/dz in order to evaluate integrals (8.21). Let us now describe the evaluation of these quantities.

To this end, let us set

$$\begin{aligned}
c_j &= \cos\left[\frac{2j-2}{4N+2}\,\pi\right],\\
\rho_j &= \{(z-z')^2+\rho^2+f^2-2f\rho c_j\}^{1/2},\\
\alpha_j &= \frac{\partial\rho_j}{\partial z} = \frac{z-z'}{\rho_j},\\
\beta_j &= \frac{\partial\rho_j}{\partial f} = \frac{f-\rho c_j}{\rho_j},\\
\gamma_j &= \frac{d\rho_j}{dz} = \alpha_j + f'\beta_j,\\
\delta_j &= \frac{d}{dz}\,\alpha_j = \frac{1-\alpha_j\gamma_j}{\rho_j},\\
\varepsilon_j &= \frac{d}{dz}\,\beta_j = \frac{f'-\beta_j\gamma_j}{\rho_j},\\
\omega_j &= \frac{ik_0}{\rho_j} - \frac{1}{\rho_j^2},\\
\theta_j &= \frac{-ik_0}{\rho_j^2} + \frac{2}{\rho_j^3},\\
G_j^* &= \frac{e^{ik_0\rho_j}}{\rho_j}.
\end{aligned} \tag{8.28}$$

The relations

$$R \equiv \{(z-z')^2+\rho^2+f^2-2f\rho\cos\theta\}^{1/2},$$

$$G^* \equiv \frac{e^{ik_0R}}{R},$$

$$G_m = \frac{1}{4\pi^2}\int_0^\pi G^*\cos m\theta\,d\theta,$$

$$\frac{dG_m}{dz} = \frac{1}{4\pi^2}\int_0^\pi\left(\frac{ik_0}{R}-\frac{1}{R^2}\right)\frac{dR}{dz}\,G^*\cos m\theta\,d\theta,$$

$$G_m^f = \frac{\partial G_m}{\partial f} = \frac{1}{4\pi^2}\int_0^\pi\left(\frac{ik_0}{R}-\frac{1}{R^2}\right)R^fG^*\cos m\theta\,d\theta,$$

$$G_m^z = \frac{\partial G_m}{\partial z} = \frac{1}{4\pi^2}\int_0^\pi\left(\frac{ik_0}{R}-\frac{1}{R^2}\right)R^zG^*\cos m\theta\,d\theta,$$

$$\frac{d}{dz} G^f_m = \frac{1}{4\pi^2} \int_0^\pi \left\{ \left(-\frac{ik_0}{R^2} + \frac{2}{R^3} \right) R^f \frac{dR}{dz} + \left(\frac{ik_0}{R} - \frac{1}{R^2} \right) \frac{d}{dz} R^f + \left(\frac{ik_0}{R} - \frac{1}{R^2} \right)^2 R^f \frac{dR}{dz} \right\} G^* \cos m\theta \, d\theta,$$

$$\frac{d}{dz} G^z_m = \frac{1}{4\pi^2} \int_0^\pi \left\{ \left(-\frac{ik_0}{R^2} + \frac{2}{R^3} \right) R^z \frac{dR}{dz} + \left(\frac{ik_0}{R} - \frac{1}{R^2} \right) \frac{d}{dz} R^z + \left(\frac{ik_0}{R} - \frac{1}{R^2} \right)^2 R^z \frac{dR}{dz} \right\} G^* \cos m\theta \, d\theta,$$

$$\cos m\theta = 2 \cos \theta \cos (m-1)\theta - \cos (m-2)\theta \tag{8.29}$$

then show that given z, z' and ρ, we can evaluate the six quantities referred to in the title of his section, for $m = 0, 1, \ldots, M+1$ by means of the following algorithm.

Algorithm 8.1—*Evaluation of* G_m, G^f_m, G^z_m, G'_m, $(G^f_m)'$, $(G^z_m)'$

1. Evaluate each of the quantities (8.28) for $j = 1, 2, \ldots, 2N+2$ as well as $f = f(z)$ and $f' = f'(z)$.
2. $(G_m, G^f_m, G^z_m, (G_m)', (G^f_m)', (G^z_m)') \leftarrow (0, 0, 0, 0, 0, 0)$.

$j \leftarrow 1$

$d_0 \leftarrow 1$

$\boxed{j \leftarrow j+1}$

$d_1 \leftarrow c_j$

$d_m \leftarrow 2c_j d_{m-1} - d_{m-2}$, $m = 2, 3, \ldots, M+1$

$$\left.\begin{array}{l}
G_m \leftarrow G_m + G_j^* d_m \\
G^f_m \leftarrow G^f_m + \omega_j \beta_j G_j^* d_m \\
G^z_m \leftarrow G^{z'}_m, + \omega_j \alpha_j G_j^* d_m \\
G'_m \leftarrow G'_m + \omega_j \gamma_j G_j^* d_m \\
(G^f_m)' \leftarrow (G^f_m)' + [(\theta_j + \omega_j^2) \beta_j \gamma_j + \omega_j \varepsilon_j] \, G_j^* d_m \\
(G^z_m)' \leftarrow (G^{z'}_m)' + [(\theta_j + \omega_j^2) \alpha_j \gamma_j + \omega_j \delta_j] \, G_j^* d_m
\end{array}\right\} m = 0, 1, \ldots, M+1.$$

$j : 2N+2$

$(<)$

$(=)$ $(G_m, G^f_m, G^z_m, G'_m, (G^f_m)', (G^z_m)')$

$$\leftarrow \frac{1}{4\pi(2N+2)} (G_m, G^f_m, G^z_m, G'_m, (G^f_m)', (G^z_m)')$$

9. Convergence

The proof of the convergence of the approximation scheme presented in Sections 6 to 8 is quite simple, using Theorem A.1 and the results of Section 5.

Let us denote the right-hand side of (A.36) by $A\bar{J}_1$ and, for $F \in H(d, d')$, let us denote l_{MN} by $P_{MN}(F)$, where $l_{MN}(z, \varphi)$ is defined in (5.5).

In view of Theorem A.1, and Section 5, we have

(a) $\|P_{MN}AJ - AJ\|_H \to 0$ for every $J \in H(d, d')$;

(b) $\|P_{MN}J^0 - \bar{J}^0\|_H \to 0$;

(c) $\sup \|P_{MN}\| \leqslant 1 + \|I - P_{MN}\| < \infty$.

Hence according to [10, pp. 469–470] it follows that the approximation $\bar{\mathscr{J}} = \bar{\mathscr{J}}_{MN}$ produced by the algorithm of Sections 6 to 8 converges to the solution $\bar{J}$ of Eq. (7.2). Moreover, by taking $N = M^2$, it follows that

$$\|\bar{J} - \bar{\mathscr{J}}_{MN}\|_H = O(e^{-\gamma N^{1/2}}) \tag{9.1}$$

for some $\gamma > 0$. Due to the quadratures and matrix solution involved in the actual algorithm we actually compute a perturbed solution $\tilde{\mathscr{J}}_{MN}$; however, due to the accuracy of the quadrature schemes described in Section 5, and since the resulting Galerkin matrix is not ill-conditioned, we also have

$$\|\bar{\mathscr{J}}_{MN} - \tilde{\mathscr{J}}_{MN}\|_H = O(e^{-\gamma N^{1/2}}), \qquad N \to \infty. \tag{9.2}$$

By combining (9.1) and (9.2), it thus follows that

$$\|\bar{J} - \tilde{\mathscr{J}}_{MN}\|_H = O(e^{-\gamma N^{1/2}}), \qquad N \to \infty. \tag{9.3}$$

Finally, in computing the scattered field $\bar{\mathscr{E}}^s = \bar{\mathscr{E}}^s_{MN}$, as described in Section 8, we similarly have, by Theorem 5.2, that

$$\|\bar{E}^s - \tilde{\mathscr{E}}_{MN}\|_H = O(e^{-\gamma N^{1/2}}), \tag{9.4}$$

where $\tilde{\mathscr{E}}_{MN}$ denotes the perturbed scattered field that we actually computed.

Appendix A: The Functions G_m, $\partial G_m/\partial f$ and $\partial G_m/\partial z$.

In this Appendix we study the functions G_m, $G_m^f = \partial G_m/\partial f$ and $G_m^z = \partial G_m/\partial z$ which appear in Section 7 to 10. The results of this study will enable us to deduce the following:

(i) Each component of the solution $\bar{J}$ of Eq. (7.2) is in $H(d, d')$.

(ii) If $m \geqslant 0$,

$$G_m(z,z') = O(|f(z)|^m),$$
$$G_m^f(z,z') = \begin{cases} O(1) & \text{if} \quad m = 0, \\ O(|f(z)|^{m-1}), & m > 0, \end{cases} \qquad \text{as} \quad z \to \pm 1, z' \in (-1,1);$$
$$G_m^z(z,z') = O(|f(z)|^m) \tag{A.1}$$

and

$$G_m(z,z') \sim \frac{1}{4\pi^2 f(z')} \log \frac{1}{|z-z'|},$$
$$G_m^f(z,z') \sim \frac{-f'(z')\,\alpha(z')^2}{4\pi^2 f(z')(z-z')}, \qquad \text{as} \quad z \to z', z' \in (-1,1).$$
$$G_m^z(z,z') \sim \frac{-\alpha(z')^2}{4\pi^2 f(z')(z-z')} \tag{A.2}$$

Results (A.1) proved to be useful for choosing the basis functions, in order to be able to obtain a convergent Galerkin method, while results (A.2) enabled us to choose the proper numerical integration technique for evaluating the singular integrals to get the coefficients P^{mn}, Q^{mn}, R^{mn} and S^{mn} in Sections 7–8.

Throughout this Appendix, the following notation is used:

$$\begin{aligned} f &= f(z), \qquad f^* = f(z'), \\ \nu &= \{(z-z')^2 + (f+f^*)^2\}^{1/2}, \\ \kappa &= \frac{4ff^*}{\nu^2}, \\ 1-\kappa &= \frac{(f-f^*)^2 + (z-z')^2}{\nu^2}. \end{aligned} \tag{A.3}$$

(a) *The functions G_m*

The functions G_m(see Eqs. (7.16) and (7.17)) are defined given z, z' on $(-1, 1)$ by

$$G_m = G_m(z,z') = \frac{1}{4\pi^2} \int_0^\pi \frac{e^{ik_0R}}{R} \cos(m\theta)\, d\theta, \tag{A.4}$$

where

$$R = \{(z-z')^2 + f^2 + f^{*2} - 2ff^* \cos\theta\}^{1/2}. \tag{A.5}$$

In terms of (A.3), we therefore have

$$R = v\left\{1 - \kappa\cos^2\frac{\theta}{2}\right\}^{1/2},$$
$$G_m = \frac{1}{2\pi^2}\int_0^{\pi/2} \frac{e^{ik_0 v\{1-\kappa\cos^2\theta\}^{1/2}}}{v\{1-\kappa\cos^2\theta\}^{1/2}}\cos(2m\theta)\,d\theta. \tag{A.6}$$

Expansion of the exponential in (A.6) and termwise integration yields

$$G_m = \frac{1}{2\pi^2}\sum_{s=0}^{\infty}\frac{(-1)^s v^{2s}k_0^{2s}}{(2s)!}\left\{\frac{J_m^s}{v} + i\,\frac{k_0 K_m^s}{2s+1}\right\}, \tag{A.7}$$

where

$$J_m^s = \int_0^{\pi/2}\{1-\kappa\cos^2\theta\}^{s-1/2}\cos(2m\theta)\,d\theta,$$
$$K_m^s = \int_0^{\pi/2}\{1-\kappa\cos^2\theta\}^{s}\cos(2m\theta)\,d\theta. \tag{A.8}$$

The relationship

$$\{1-\kappa\cos^2\theta\}^a\cos(2m\theta)$$
$$= \{1-\kappa\cos^2\theta\}^{a-1}$$
$$\times\left[\left(1-\frac{\kappa}{2}\right)\cos(2m\theta) - \frac{\kappa}{4}\cos(2m+2)\theta - \frac{\kappa}{4}\cos(2m-2)\theta\right] \tag{A.9}$$

shows that

$$J_m^s = \left(1-\frac{\kappa}{2}\right)J_m^{s-1} - \frac{\kappa}{4}J_{m+1}^{s-1} - \frac{\kappa}{4}J_{m-1}^{s-1},$$
$$K_m^s = \left(1-\frac{\kappa}{2}\right)K_m^{s-1} - \frac{\kappa}{4}K_{m+1}^{s-1} - \frac{\kappa}{4}K_{m-1}^{s-1}, \tag{A.10}$$

whereas, by (A.8),

$$J_{-m}^s = J_m^s, \qquad K_{-m}^s = K_m^s. \tag{A.11}$$

Relationships (A.10) show that in order to evaluate G_m using (A.7), we need only know J_m^0 and K_m^0, for $m = 0, \pm 1, \pm 2, \ldots$; we can then get the remaining J_m^s and K_m^s for $s > 0$ using (A.10). To this end, integrating termwise (A.8) and using the identity

$$\int_0^{\pi/2}\cos^{2n}\theta\cos(2m\theta)\,d\theta = \binom{2n}{n+m}\frac{\pi}{2^{2n+1}} \tag{A.12}$$

we get

$$J_m^0 = \frac{\pi}{2} \frac{(\frac{1}{2})_m \kappa^m}{2^{2m} m!} F\left(\frac{1}{2} + m, \frac{1}{2} + m; 1 + 2m; \kappa\right),$$
$$K_m^0 = \pi/2 \quad \text{if} \quad m = 0,$$
$$= 0 \quad \text{if} \quad m \neq 0. \tag{A.13}$$

In (A.13), F denotes the hypergeometric function. Thus we may compute J_m^0 for $0 \leqslant \kappa \leqslant 0.6$ by means of the formula

$$J_m^0 = \frac{\pi}{2} \frac{(\frac{1}{2})_m \kappa^m}{2^{2m} m!} \sum_{n=0}^{\infty} \frac{[(\frac{1}{2} + m)_n]^2}{(1 + 2m)_n \, n!} \kappa^n \tag{A.14}$$

whereas, if $0.6 < \kappa < 1$ (see [1, p. 559]) it is preferable to use the formula

$$J_m^0 = \kappa^m \sum_{n=0}^{\infty} \frac{[(\frac{1}{2} + m)_n]^2}{[n!]^2} \times \left[\psi(n+1) - \psi\left(\frac{1}{2} + m + n\right) - \frac{1}{2}\ln(1-\kappa)\right] (1-\kappa)^n, \tag{A.15}$$

where

$$\psi(1) = -\gamma = -0.5772156649,$$
$$\psi(\tfrac{1}{2}) = -\gamma - 2\ln 2 = -1.963510026, \tag{A.16}$$
$$\psi(z+1) = 1/z + \psi(z).$$

Equation (A.15) shows that

$$J_m^0 \sim -\tfrac{1}{2}\ln(1-\kappa) \qquad \text{as} \quad \kappa \to 1^-. \tag{A.17}$$

Now, by (A.3)

$$1 - \kappa = \frac{\nu^2 - 4ff^*}{\nu^2} = \frac{(z-z')^2 + (f - f^*)^2}{\nu^2}$$
$$\sim \frac{[1 + f^{*\prime 2}](z-z')^2}{4f^{*2}} \qquad \text{as} \quad z \to z' \tag{A.18}$$

so that, by combining (A.17) and (A.18), we get

$$J_m^0 \sim \ln\left[\frac{2f^*}{(1 + f^{*\prime 2})^{1/2}} \frac{1}{|z - z'|}\right], \qquad z \to z'. \tag{A.19}$$

Using induction on (A.10), it thus follows that

$$J_m^s \sim 0 \qquad \text{as} \quad z \to z', \quad s > 0. \tag{A.20}$$

In view of (A.7), (A.13), (A.19) and (A.20) it therefore follows that

$$G_m \sim \frac{1}{2\pi^2} \frac{J_m^0}{v} \sim \frac{1}{4\pi^2 f^*} \ln \left[\frac{2f^*}{(1+f^{*\prime 2})^{1/2}} \frac{1}{|z-z'|} \right] \qquad \text{as} \quad z \to z', \tag{A.21}$$

which is the first of (A.2).

In view of (A.3), it follows that

$$\kappa = O(f) \qquad \text{as} \quad z \to \pm 1, \tag{A.22}$$

for all $z' \in [-1, 1]$. Hence by (A.13) and (A.14),

$$G_m = O(f^{|m|}) \qquad \text{as} \quad z \to \pm 1, \qquad \text{for all } z \in [-1, 1]. \tag{A.23}$$

(b) *The functions* G_m^f, $G_m^{z'}$

Upon differentiating expressions (A.3) we get

$$\begin{aligned} v^f &\equiv \frac{\partial v}{\partial f} = \frac{f+f^*}{v}, \\ v^z &\equiv \frac{\partial v}{\partial z} = \frac{z-z'}{v}, \\ \kappa^f &= \frac{\partial \kappa}{\partial f} = \frac{4f^*}{v^2} - \frac{8ff^*(f+f^*)}{v^4}, \\ \kappa^z &= \frac{\partial \kappa}{\partial z} = -\frac{8ff^*}{v^3} \frac{(z-z')}{v}. \end{aligned} \tag{A.24}$$

We therefore note that

$$\begin{aligned} 2v^f + v\frac{\kappa^f}{\kappa} &= \frac{v}{f}, \\ 2v^z + v\frac{\kappa^z}{\kappa} &= 0. \end{aligned} \tag{A.25}$$

By means of these solutions as well as (A.10), we get the identities

$$G_m^f \equiv \frac{\partial G_m}{\partial f} = \frac{1}{2\pi^2} \sum_{s=0}^{\infty} \left\{ \frac{(-1)^s \nu^{2s} k_0^{2s}}{(2s)!} \left[\frac{s-\frac{1}{2}}{\nu} \left(\frac{1}{f} J_m^s - \frac{\kappa^f}{\kappa} J_m^{s-1} \right) \right. \right. \\ \left. \left. \times \frac{-i\nu^2 k_0^3}{2(2s+1)(2s+3)} \left(\frac{1}{f} K_m^{s+1} - \frac{\kappa^f}{\kappa} K_m^s \right) \right] \right\} \tag{A.26}$$

and

$$G_m^z = \frac{z-z'}{\pi^2 \nu^3} \sum_{s=0}^{\infty} \left\{ \frac{(-1)^s \nu^{2s} k_0^{2s}}{(2s)!} \right. \\ \left. \times \left[\left(s - \frac{1}{2} \right) J_m^{s-1} + \frac{-i\nu^2 k_0^3}{2(2s+1)(2s+3)} K_m^s \right] \right\}. \tag{A.27}$$

These series may be readily computed using the identities

$$\begin{aligned} J_m^{-1} &= \frac{\pi}{2} \frac{(\frac{3}{2})_m \kappa^m}{2^{2m} m!} F\left(\frac{3}{2} + m; \frac{1}{2} + m; 2m+1; \kappa \right) \\ &= \frac{\pi}{2} \frac{(\frac{3}{2})_m \kappa^m}{2^{2m} m!} \sum_{n=0}^{\infty} \frac{(\frac{3}{2}+m)_n (\frac{1}{2}+m)}{(2m+1)_n \, n!} \kappa^n, \qquad 0 \leqslant \kappa \leqslant 0.6, \\ &= \frac{\kappa^m}{1-\kappa} + \kappa^m \left(m^2 - \frac{1}{4} \right) \sum_{n=0}^{\infty} \frac{(\frac{3}{2}+m)_n (\frac{1}{2}+m)_n}{n! \, (n+1)!} (1-\kappa)^n \\ &\quad \cdot \left[\ln(1-\kappa) + 2\psi\left(\frac{1}{2}+m+n \right) - 2\psi(n+1) + \frac{1}{m+n+\frac{1}{2}} - \frac{1}{n+1} \right] \\ &\qquad \text{if} \quad 0.6 < \kappa < 1 \end{aligned} \tag{A.28}$$

along with (A.10). These identities were obtained via a procedure similar to that used to get (A.14) and (A.15).

Expressions (A.26) to (A.28) enable us to deduce various growth properties of G_m^f and G_m^z as $z \to z'$ and as $z \to \pm 1$.

By (A.28) and (A.3),

$$\begin{aligned} J_m^{-1} &\sim \frac{\kappa^m}{1-\kappa} \\ &\sim \frac{4 f^{*2} \alpha^{*2}}{(z-z')^2} \qquad \text{as} \quad z \to z' \in (-1, 1) \end{aligned} \tag{A.29}$$

and

$$J_m^{-1} = O(f^m) \qquad \text{as} \quad z \to \pm 1. \tag{A.30}$$

Combining these results with the asymptotic identities

$$\begin{aligned}\frac{\kappa^f}{\kappa} &\sim -\frac{f^{*\prime}(z-z')}{2f^{*2}}, \qquad z \to z', \\ &\sim \frac{1}{f}, \qquad z \to \pm 1,\end{aligned} \tag{A.31}$$

and

$$\begin{aligned}\frac{\kappa^z}{\kappa} &\sim -\frac{2f^{*2}(z-z')}{v^2}, \qquad z \to z', \\ \frac{\kappa^z}{\kappa} &= O(f), \qquad z \to \pm 1,\end{aligned} \tag{A.32}$$

we get

$$\begin{aligned}G_m^f &\sim -\frac{1}{4\pi^2}\frac{f^{*\prime}\alpha^{*2}}{f^*(z-z')}, \qquad z \to z', \\ G_m^z &\sim -\frac{1}{4\pi^2}\frac{\alpha^{*2}}{f^*(z-z')}, \qquad z \to z'.\end{aligned} \tag{A.33}$$

Relation (A.10), (A.13), (A.26), (A.28) and (A.31) yield

$$\begin{aligned}G_m^f &= O(1) \qquad \text{if} \quad m=0 \\ &= O(f^{m-1}) \qquad \text{if} \quad m>0\end{aligned} \qquad \text{as} \quad z \to \pm 1, \tag{A.34}$$

while relations (A.10), (A.13), (A.27), (A.28) and (A.32) yield

$$G_m^z = O(f^m) \qquad \text{as} \quad z \to \pm 1. \tag{A.35}$$

(c) *Analyticity of* $\bar{K}$

The above results show that

(i) G_m is bounded as a function of z on $[-1, 1]$, except at $z = z'$, where it becomes unbounded according to (A.21);

(ii) $G_m(z, z')$ is an analytic function of $z \in \Omega_d$, except at $z = z'$, where it has a singularity of the form (A.21).

In the following theorem $H(d, d')$ is defined as in Section 5.

THEOREM A.1. *Let* $\bar{K}$ *be the solution of Eq.* (4.6), *and let* $\bar{J}$ *be defined by* (7.1). *Then each component of* $\bar{J}$ *is in* $H(d, d')$.

Proof. Each component of the incident field $\bar{J}^0$ (see Eq. (7.3)) is in $H(d, d')$, where $d' > 0$ is arbitrary. In view of Eq. (7.2), we need only show that if $\bar{J}_1 = (J_{1t}, J_{1\varphi})$ denotes a pair of functions in $H(d, d')$, each of which is bounded on each of the sets

$$S_1 = \{(z, \omega): z \in \Omega_d, |\omega| = 1\},$$
$$S_2 = \{(z, \omega): -1 \leqslant z \leqslant 1, 1/d' \leqslant |\omega| \leqslant d'\},$$

where $d' > 1$ is arbitrary, then each of the components of

$$\begin{aligned} J_2 &= (J_{2t}, J_{2\varphi}) \\ &= \left\{ v \int_S \left[\frac{i\omega\mu}{v} \bar{J}_1 G + \eta \frac{Z}{v} (\hat{n} \times \bar{J}_1) \times \nabla G \right.\right. \\ &\qquad \left.\left. + \frac{1}{i\omega\varepsilon} \nabla \cdot \left(\frac{1}{v} \bar{J}_1 \right) \nabla G \right] dS \right\}_{\tan} \end{aligned} \tag{A.36}$$

is in $H(d, d')$.

By our assumption on f in Section 3, the solution $\bar{K} = (K_t, K_\varphi)$ of Eq. (4.2) is bounded on $\mathscr{S} = [-1, 1] \times [0, 2\pi]$. Hence for $z \in [-1, 1]$, each component of $\bar{J}_1$ has the form

$$F(z, e^{i\varphi}) = \sum_{-\infty}^{\infty} a_m(z)\, e^{im\varphi}, \tag{A.37}$$

where $a_m/v \in H(\Omega_d)$; substituting this form of an expression into (A.36) for J_{1t} and $J_{1\varphi}$ and noting that $a_m/v = O(e^{-d'|m|})$ $\forall z \in [-1, 1]$ and for all $d' > 0$, we deduce, by inspection of (7.27)–(7.30) and (A.1) and (A.2) that $|P^{mn}|$, $|Q^{mn}|$, $|R^{mn}|$ and $|S^{mn}|$ are $O(e^{-d'|m|})$ for all $d' > 0$ and for all $z' \in [-1, 1]$. That is, $F(z, \omega) \in H(A_{d'})$ as a function of ω, for all $z \in [-1, 1]$.

Hence in order to complete the proof, we need only show that if $\theta_n/v \in H(\Omega_d)$, then each of the right-hand sides of (7.27)–(7.30) is in $H(\Omega_d)$.

In view of the results of parts (a) and (b) of Appendix A, the coefficients of θ_n/v in the integrals (7.27)–(7.30) are of the following three types:

$$a(z, z'), \qquad a(z, z') \log|z - z'|, \qquad a(z, z')/(z - z'),$$

where $a(z, z')$ is a bounded function in $H(\Omega_d) \times H(\Omega_d)$. Hence, we need only show that given $g \in H(\Omega_d)$, each of the functions g_1, g_2 and g_3 are analytic in Ω_d, where

$$g_1(z') = \int_{-1}^{1} a(z, z')\, g(z)\, dz;$$

$$g_2(z') = \int_{-1}^{1} a(z, z') \log |z - z'|\, g(z)\, dz; \tag{A.38}$$

$$g_3(z') = P.V. \int_{-1}^{1} \frac{a(z, z')}{z - z'}\, g(z)\, dz.$$

It is obvious that $g_1 \in H(\Omega_d)$.
Next if $z' \in \Omega_d \cap \{\operatorname{Im} z' > 0\}$ then [11]

$$g_3^*(z') = \int_{-1}^{1} \frac{a(z, z')}{z - z'}\, g(z)\, dz \tag{A.39}$$

is analytic in this region, and indeed, by altering the path of integration in (A.39) to the lower boundary of Ω_d, we see that g_3^* is in fact analytic in Ω_d. If we now return the path of integration to the interval $(-1, 1)$ and let $\operatorname{Im} z' \to 0$, we find that for $z' \in (-1, 1)$,

$$g_3^*(z') = \pi i a(z', z')\, g(z') + \pi i g_3(z'); \tag{A.4}$$

this expression shows that since both $g_3^*(z')$ and $a(z', z')$ have an analytic extension into Ω_d, so does g_3. Hence g_3 is analytic in Ω_d.

Finally, writing g_2 in the form of a convergent sum

$$g_2(z') = \int_{-1}^{1} \sum_k \alpha_k(z)\, \beta_k(z') \log |z - z'|\, dz, \tag{A.41}$$

where the functions α_n and β_n are in $H(\Omega_d)$, we need only show that g_k^* is analytic in Ω_d, where

$$g_k^*(z') = \int_{-1}^{1} \alpha_k(z) \log |z - z'|\, dz. \tag{A.42}$$

Upon differentiating this expression *carefully*, we see that

$$g_k^{*\prime}(z') = P.V. \int_{-1}^{1} \frac{\alpha_k(z)}{z - z'}\, dz. \tag{A.43}$$

By our argument involving g_3 above, it follows that $g_k^{*\prime}$ is analytic in Ω_d, i.e., g_k^* and hence g_2 is analytic in Ω_d. This completes the proof of Theorem A.1.

By assumption for the case of finite conductivity of the body B, the function f is such that the surface S satisfies Liapunov conditions, in which case the surface current $\bar{K}$ is bounded on S. However, one notes that integrals (7.27)–(7.30) converge so long as $f|\theta_n/\nu| = o(1)$ as $z \to \pm 1$, i.e., so long as $fK = o(1)$ as $z \to \pm 1$. Thus our method gives answers even if S is cone-shaped at one or both ends, although the Liapunov conditions are then violated, and our results may have no physical significance.

References

1. A. Abromowitz and I. A. Stegun, "Handbook of Mathematical Functions," N.B.S. Applied Math. Series No. 53, Nat. Bur. Standards, Washington D.C., 1964.
2. M. G. Andreson, Scattering for bodies of revolution by an exact method, *IEEE Trans. Antennas Propagation* **AP-13** (1965), 303–310.
3. P. Barber and C. Yeh, Scattering of Electromagnetic Waves by Arbitrarily Dielectric Bodies, *Appl. Opt.* **14** (1975), 2864–2872.
4. S. L. Borison, "Diagonal Representation of the Radar Scattering Matrix for an Axially Symmetric Body," Mass. Inst. Tech. Lincoln Lab, 41L-0031, 1964.
5. D. Castellanos, "Notes on Electromagnetic Scattering from Rotationally Symmetric Bodies with an Impedence Boundary Condition," The University of Michigan College of Engineering, Department of Electrical Engineering Radiation Lab, TR No. 1, 1966.
6. M. Dubois and J. C. Lacket, The integral formulation of boundary value problems, *Int. J. Solids Struct.* **11** (1975), 9/89–9/109.
7. D. Greenspan and P. Werner, A numerical method for the exterior Dirichlet problem for the reduced wave equation, *Arch. Rational Mech. Anal.* **23** (1966), 288–316.
8. J. K. Hayes, D. K. Kahaner, and R. Kellner, A numerical comparison of integral equations of the first and second kind for conformal mapping, *Math. Comput.* **29** (1975), 512–521.
9. G. W. Hohmann, Three-dimensional induced polarization and electromagnetic scattering, *Geophysics* **40** (1975), 309–324.
10. Y. Ikebe, The Galerkin method for the numerical solution of Fredholm integral equations of the second kind, *SIAM Rev.* **14** (1972), 465–491.
11. Y. Ikebe, T. Y. Li, and F. Stenger, Numerical solution of the Hilbert problem, *in* "Proceedings, 1975 Univ. of Calgary Conference on Approximation Theory," Academic Press, New York, 1976.
12. Y. Ikebe, M. S. Lynn, and W. P. Timlake, The numerical solution of the integral equation formulation of the single interface Neumann problem, *SIAM J. Numer. Anal.* **6** (1969), 334–346.
13. E. M. Kennaugh, "Multipole Field Expansions and Their Use in Approximate Solutions of Electromagnetic Scattering Problems," Ph. D. dissertation, Department of Electrical Engineering, The Ohio State University 1959.
14. R. G. Kongoumjian, Backscattering from a cylindrical loop, *Appl. Sci. Res. B***16** (1957), 165–179.
15. M. A. Leontovich, "Investigation of Propagation of Radio Waves, Part II," Moscow, 1948.
16. L. Lundin, and F. Stenger, Cardinal type approximation of a function and its derivations, *SIAM J. Math. Anal.* **10** (1979), 139–160.
17. S. G. Mikhlin, Linear Equations of Mathematical Physics, Holt Rinehart & Winston, New York, 1967.
18. J. H. Richmond, Digital computer solutions of the rigorous equations for scattering problems, *Proc.* IEEE **53** (1965), 796–804.

19. F. V. SCHULTZ, G. M. RUCKGABER, S. RICHTER, AND J. K. SCHINDLER, "The Theoretical and Numerical Determination of Radar Cross Section of a Finite Cone," School of Electrical Engineering, Purdue University, Lafayette, Ind., TR-EE 64-14, Doc AD 606 828, 1964.
20. P. SCHWEITZER, "Electromagnetic Scattering from Rotationally Symmetric Perfect Conductors," Mass. Inst. Tech. Lincoln Lab., Group 22 Project Report PA-88 (BMRS), 1965.
21. T. B. A. SENIOR, Impedence boundary conditions for imperfectly conducting surfaces, *Appl. Sci. Res. B* **8** (1960), 418–436.
22. F. STENGER, Quadrature formulae based on the trapezoidal formula, *J. Inst. Math. Appl.* **12** (1973), 103–114.
23. F. STENGER, Approximation via Whittaker's cardinal function, *J. Approximation Theory* **17** (1976), 222–240.
24. F. STENGER, Kronecker product extensions of linear operations, *SIAM J. Numer. Anal.* **5** (1968), 422–435.
25. F. STENGER, Optimal upper and lower estimates on the rate of convergence of minimum norm approximations, in Hp, *Bull. Amer. Math. Soc.* in press.
26. J. A. STRATTON, "Electromagnetic Theory," McGraw–Hill, New York, 1941.
27. P. C. WATERMAN, Scattering by dielectric obstacles, *Alta Freq.* **38** (Speciale) (1969), 348–352.